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On the property structure of realist collapse interpretations of quantum mechanics and the so-called "counting anomaly"

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Abstract *The aim of this article is twofold. Recently, Lewis has presented an argument, now known as the "counting anomaly", that the spontaneous localization approach to quantum mechanics, suggested by Ghirardi, Rimini, and Weber, implies that arithmetic does not apply to ordinary macroscopic objects. I will take this argument as the starting point for a discussion of the property structure of realist collapse interpretations of quantum mechanics in general. At the end of this I present a proof of the fact that the composition principle, which holds true in standard quantum mechanics, fails in all realist collapse interpretations. On the basis of this result I reconsider the counting anomaly and show that what lies at the heart of the anomaly is the failure to appreciate the peculiarities of the property structure of such interpretations. Once this flaw is uncovered, the anomaly vanishes.*

1. Introduction: collapse interpretations, tails, and the counting anomaly

Lewis (1997) considers a marble and a box. The marble has two states, namely $|\psi_{\text{in}}\rangle$ (the marble is inside the box) and $|\psi_{\text{out}}\rangle$ (the marble is outside the box). These states are mutually exclusive and therefore orthogonal; that is, $\langle\psi_{\text{in}}|\psi_{\text{out}}\rangle = 0$. Furthermore, consider a measurement device $\hat{B}$, measuring whether the marble is inside or outside the box. Quantum mechanics (QM) has it that not only eigenstates of $\hat{B}$, $|\psi_{\text{in}}\rangle$ and $|\psi_{\text{out}}\rangle$, but any superposition $|\psi_{\text{m}}\rangle = a|\psi_{\text{in}}\rangle + b|\psi_{\text{out}}\rangle$ of these can be the state of the marble system (where a and b are arbitrary complex numbers satisfying $|a|^2 + |b|^2 = 1$). But what are the physical properties of a system in such a state? The answer to this question obviously depends on how the connection between quantum states and physical properties is construed. The standard way to relate quantum states and properties is the *Eigenstate–Eigenvalue Rule* ("E–E rule" henceforth).[1]

> An observable $\hat{O}$ has a well-defined value for a quantum system S in state $|\psi\rangle$ if, and only if, $|\psi\rangle$ is an eigenstate of $\hat{O}$.

Since $|\psi_{\text{m}}\rangle$ is not an eigenstate of $\hat{B}$, it defies interpretation on the basis of the E–E rule and the marble is neither inside nor outside the box. But this conclusion is obviously unacceptable, since our experience indicates that the marble has a definite location.

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Reconciling everyday experience with this unwelcome consequence of the quantum formalism is the infamous measurement problem.

In an attempt to overcome this difficulty, von Neumann (1955) postulated that whenever a measurement is performed on the system its state instantaneously collapses into one of the eigenstates of the measured observable. What we are left with then is a state that can be interpreted on the basis of the E–E rule without any difficulty. However, although the collapse postulate restores the interpretability of the post-measurement state in terms of the E–E rule, it turns out that it raises more problems than it solves. What defines a measurement? At what stage of the measurement process does the collapse take place (trigger problem)? Why should the properties of a system depend on actions of observers or, even worse, on their minds in the first place?

An ingenious way to overcome these difficulties has been suggested by Ghirardi *et al.* (1986) and has been put in a particularly elegant and simple form by Bell (1987). It has become customary to refer to this account as "GRW theory". Its leading idea is to evade the above-mentioned problems by reformulating collapse interpretations in a way that avoids appeal to observers. This is achieved by no longer considering collapses as measurement-induced and making them an integral part of what happens in nature; collapses "just happen" at random in nature and do not in any way depend on observers. To be more precise, GRW theory postulates that in an N-particle system a collapse occurs once in τ/N s, where τ is a new constant of nature (which, according to GRW, is of the order of 10^{15} s). To flesh this basic idea out, GRW theory provides a well-defined collapse mechanism, but since the details do not matter for what follows I will not dwell on them here.

Unfortunately this is not the end of the story yet. Collapses have been introduced to ensure that the system is in an eigenstate of some observable, $\hat{B}$ for instance, at the conclusion of a measurement, but upon closer examination it turns out that this is exactly what they generally cannot achieve. A collapse can leave the system in a proper eigenstate only if the basis is discrete. In the case of continuous observables, such as position, this is not possible. (This point is directly relevant to the above example, since measuring whether a marble is in the box amounts to measuring its position.) There are three independent reasons why a collapse to a position eigenstate, say, is unattainable. First, as a consequence of the uncertainty relation, the more localized a wave function is in position space, the higher its dispersion in momentum space becomes, and the more energy the system can possess after a collapse. Thus, if we allow for strongly localizing collapses, the system could spontaneously heat up (Clifton & Monton, 1999, p. 698). However, such spontaneous heating has never been observed. Therefore, a collapse cannot render the wave function too narrow without contradicting experimental facts. Second, it is by now a well-known property of QM that a wave function which, at a certain instant, lacks tails (i.e. has no parts that extend to infinity) will always instantaneously grow them back. Hence, even if a strongly localizing collapse were allowed to occur, an instant later we would be back where we started. Third, the position eigenstate $|x\rangle$ is not even an element of the (separable) Hilbert space which is the state space of the system. To accommodate states like $|x\rangle$ one would have to move to a formulation of QM based on a rigged Hilbert space, and it is still controversial whether this is the right move.

As a consequence of this, a system's wave function cannot be arbitrarily narrow after a collapse. At the conclusion of a reduction process, we find the system in a state exhibiting tails. The GRW theory does justice to this limitation since a GRW-hit does not leave the marble in a precise eigenstate of the position operator but in a state that is "close" to it in the sense that it is a somehow "smeared out" eigenstate (technically speaking, the original state ψ gets multiplied by a Gaussian which makes it more localized, but it never becomes equal to a proper position eigenstate).

However, now we are back where we started. If at the conclusion of a collapse the system is not in a proper position eigenstate, the E–E rule is not applicable and we cannot tell what the location of the object is. So an alternative to the E–E rule is needed. Common physical wisdom has it that "close" is actually good enough. In order to say that a particle is located at x, it is too restrictive to require that the system's state is $|x\rangle$. Rather, it is sufficient to say that it is somewhere "within a narrow interval around x" (see for instance Sakurai, 1994, pp. 42–43). This idea has recently been rendered more precise and introduced into the philosophical literature by Albert and Loewer (1995). According to them, a particle with wave function $\psi(r)$ is located in the interval R iff the major part of $|\psi(r)|^2$ is in R; that is, iff $\int_R |\psi(r)|^2 \, \mathrm{d}r \geq 1 - \varepsilon$, where ε is a positive real number close to zero. The generalization of this rule to a system with n degrees of freedom is straightforward: the system with wave function $\psi(r_1, \ldots, r_n)$ is located in the n-dimensional interval $R_1 \times \ldots \times R_n$ iff $\int_{R_1 \times \ldots \times R_n} |\psi(r_1, \ldots, r_n)|^2 \, \mathrm{d}^n r \geq 1 - \varepsilon$. Clifton and Monton (1999) call this rule the "fuzzy link". The choice of an appropriate value for ε is a subtle issue, and I will have more to say about it later on. In what follows I will use the label "fuzzy quantum mechanics" (FQM) to refer to any interpretation of QM that takes into account the fact that collapses do not leave the system in precise (position) eigenstates and that interprets these non-eigenstates in terms of the fuzzy link—in particular, GRW theory and a realistically understood von Neumann collapse theory fall under this category.

The fuzzy link naturally gives rise to the following definition.

> Let $e_1, \ldots, e_n$ be n arbitrary entities (e.g. marbles). Then the ensemble[2] consisting of these entities, $E = \{e_1, \ldots, e_n\}$, with wave function $\psi(r_1, \ldots, r_n)$ has the property of being in the interval $R_1 \times \ldots \times R_n$ iff
>
> $$\int_{R_1 \times \ldots \times R_n} |\psi(r_1, \ldots, r_n)|^2 \, \mathrm{d}^n r \geq 1 - \varepsilon. \tag{1}$$
>
> Furthermore, let $P_{\varepsilon, R_1 \times \ldots \times R_n}(e_1, \ldots, e_n)$ be the proposition stating that E has the property of being in the interval $R_1 \times \ldots \times R_n$; this proposition is true iff equation (1) holds.

Note that E can also consist of just one object e. In this case the definition reduces to: the entity e with wave function $\psi(r)$ has the property of being in the interval R, i.e. $P_{\varepsilon, R}(e)$ is true, iff $\int_R |\psi(r)|^2 \, \mathrm{d}r \geq 1 - \varepsilon$.

Let's now see how all this bears on the marbles. For the reasons outlined above, the best we can expect is to find the system after a collapse in a highly asymmetric state of the form $|\psi_\mathrm{m}\rangle = a|\psi_\mathrm{in}\rangle + b|\psi_\mathrm{out}\rangle$ (or $|\psi_\mathrm{m}\rangle = b|\psi_\mathrm{in}\rangle + a|\psi_\mathrm{out}\rangle$ likewise) where $1 > |a| \gg |b| > 0$ and $|a|^2 + |b|^2 = 1$. According to the fuzzy link, if $|b|^2 \leq \varepsilon$ then the marble is in the box: $\int_{R_\mathrm{in}} |\psi_\mathrm{m}(r)|^2 \, \mathrm{d}r = |a|^2 \geq 1 - \varepsilon$, where R_in is the region we associate with being in the box.

So far so good. However, in his recent paper Lewis (1997) has presented an argument to the conclusion that this relaxation of the E–E rule entails that arithmetic does not apply to ordinary macroscopic objects such as marbles. This argument is now commonly referred to as the "counting anomaly" and runs as follows. Enlarge your box and place not only one but a large number n of marbles in it. Furthermore, assume that no interaction takes place between the marbles (this can be accomplished, for example, by making the box long and slim so that all marbles lie side by side without touching each other). The state of the ensemble is $|\psi_{\mathrm{total}}\rangle = |\psi_{\mathrm{m}}\rangle_1 \ldots |\psi_{\mathrm{m}}\rangle_n$.

When we now interpret $|\psi_{\mathrm{total}}\rangle$ in terms of the fuzzy link we are faced with a paradox. More specifically, we find that the ensemble consisting of the n marbles is not in the box: $\int_{R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}} |\psi_{\mathrm{total}}|^2 \ \mathrm{d}^n r = |a|^{2n}$, but $|a|^{2n} \ll 1 - \varepsilon$ since $|a|$ is smaller than 1. Hence, we make up a system of n marbles each of which *individually* is in the box and end up with an n-marble system which is not in the box. This paradox is called "counting anomaly" for the following reason. Making sure that marble 1, marble 2, ..., marble n are in the box is exactly how we count marbles (Lewis calls this the "enumeration principle"), and this means that putting one marble after the other in the box is structurally identical to the process of counting. But, as the above argument shows, by doing so we end up with a state in which it is false that the ensemble of marbles ends up being in the box. Hence counting is impossible and we must conclude that arithmetic does not apply to macroscopic objects such as marbles—that is the counting anomaly.

Finally I should stress again that although this anomaly has been presented as an argument against GRW theory in particular, the above discussion has made it clear that it equally threatens *every* interpretation of QM that falls into the category of FQM.[3] Even if we were to solve all the problems in connection with the notion of measurement (the trigger problem and so on), the counting anomaly would still await a solution.

2. Unsuccessful routes around the anomaly

This anomaly is embarrassing and calls for a solution. In this section I discuss three attempts to deal with it, all of which turn out to fail, as I shall argue. The first turns on the fact that position, as construed by the fuzzy link, is a vague notion; the second and the third are arguments put forward in a debate between Ghirardi and Bassi (1999) and Bassi and Ghirardi (1999, 2001) on one side, and Clifton and Monton (1999, 2000) on the other side. These failures suggest that deeper reflection on the problem is needed. In the following section I will prove a general theorem about the property structure of collapse interpretations and show how this theorem can be brought to bear on the anomaly.

Suppose we have a heap of sand (Sainsbury, 1995, pp. 23–24). Now we remove one grain, what remains is still a heap—removing a single grain cannot turn a heap into something that is not a heap. Nevertheless, if we keep removing one grain after the other, we end up with no grains at all, and no grains certainly do not make up a heap. So it seems that there must be a least number of grains which still makes up a pile. But what is this number? We simply may not know, or we arguably can think that it is just silly to assume that such a number even exists. But isn't location, as construed by the fuzzy link, just like the number of grains in a heap? Yes it is. How close to zero need ε be in order to have a localized state? Albert and Loewer (1995, pp. 87–92) discuss the issue of the choice of a correct value for ε at length and come to the conclusion that, apart from the obvious restriction that ε has to be larger than zero and smaller than one half, there fails to be any precise matter of fact about what the correct value of ε is.

Does this give rise to a revision of the anomaly? If there is, after all, no unique correct value for ε, is it not possible to evade the anomaly by being a bit more liberal about the admissible values of ε? Concretely, this suggestion amounts to saying that even if $\int_{R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}} |\psi_{\mathrm{total}}|^2 \ \mathrm{d}^n r < 1 - \varepsilon$, there always exists an ε' such that $\int_{R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$

$|\psi_{\text{total}}|^2\ \mathrm{d}^n r > 1 - \varepsilon'$. Since there is no one single correct value for ε, there is no reason why we should prefer ε to ε' and hence all marbles are in the box.

Unfortunately, a closer look at the actual numerical values reveals that in general this will not do. If a sufficiently large number of marbles is available, $\int_{R_{\text{in}} \times \ldots \times R_{\text{in}}} |\psi_{\text{total}}|^2\ \mathrm{d}^n r = |a|^{2n}$ can be as close to zero as we please and even on a liberal reading one then can no longer say that the n marbles are in the box (since $\varepsilon < 1/2$). Another suggestion would be to make the original ε in the one-marble fuzzy link smaller instead of the one in the n-marble fuzzy link bigger. But here we also run into trouble. By making ε smaller we get closer to a proper position eigenstate, and when ε is small enough we may require more than what is physically possible. For the reasons mentioned above, there are limits as to how close a physical state can come to a position eigenstate and ε cannot be arbitrarily small. But then we cannot evade the conclusion that, by taking a large enough number of marbles, $|a|^{2n}$ almost equals zero and the n-marble ensemble cannot possibly be inside the box.[4] The upshot of all this is that fiddling around with the value of ε does not help.

This anomaly has also been the starting point of a lively debate between Ghirardi and Bassi on one side and Clifton and Monton on the other. The scope of this debate is restricted to the discussion of the anomaly within the context of GRW theory. The remainder of this section will be devoted to a discussion of the arguments put forward in this debate. My conclusion will be that none of the lines of reasoning taken in this debate leads to a solution of the problem.

In a first reply to Lewis, Ghirardi and Bassi (1999) have argued that the alleged anomaly is not an anomaly at all and dismiss the argument as "devoid of any sense". They argue that the state $|\psi_{\text{total}}\rangle$, on which the argument turns, is not stable according to GRW dynamics and will collapse immediately to an unproblematic state. Clifton and Monton (1999) have pointed out that this is not correct. They show that even if $|\psi_{\text{total}}\rangle$ is reduced immediately, the reduced state still has tails and therefore gives rise to the same difficulty.

Clifton and Monton consider the counting anomaly to be a serious problem for the GRW theory, one that calls for a solution. For this reason, in the second part of their paper, they present a sophisticated argument for the conclusion that while the enumeration principle *can* fail, GRW theory itself ensures that this failure can never be *observed*. They point out that once the counting apparatus which records how many marbles are in the box is modelled correctly on the basis of the principles of GRW theory, i.e. once we give a correct operationalization of the counting process, the anomaly disappears. In three subsequent papers (Bassi & Ghirardi, 1999, 2001; Clifton & Monton, 2000) each of the parties defends its view but no new arguments come into play.

Where does all that leave us? Though there is no agreement as to what the correct solution of this problem is, in the end, both parties at least agree that the anomaly can be dismissed. Does that mean that the clouds over GRW have been blown away and the sky is clear again? I do not think so. On the contrary, it seems to me that notwithstanding everything that has been said so far, the problem has not been solved. There are two reasons for this. First, Ghirardi and Bassi's reply is flawed for the reasons Clifton and Monton have pointed out. I have nothing to add to their argument. But, second, Clifton and Monton's own solution does not seem satisfactory to me either.

Let's briefly recall their argument. The crucial question is whether it is sufficient or not to suppress the anomaly. Can we continue to take the theory seriously just because there is a mechanism that suppresses the manifestations of the anomaly? Clifton and Monton are quite sensitive to this question and discuss it at length in the last section of their paper (1999). They point out that "by itself" suppressing the empirical manifestations does not resolve the problem. Nevertheless, their final answer to the above question is "yes". They justify their decision as follows. Prima facie, GRW is a theory about wave functions, and nothing else. It is only once we relate these wave functions to our ordinary language via the fuzzy link that all these problems can crop up. The fuzzy link does not add anything of ontological import to the theory, but simply provides a way of mapping our "particle" language on to a theory whose fundamental language concerns wave functions. Therefore, the fuzzy link has something of "the status of a postulate that (to echo Reichenbach […]) 'is neither true nor false, but a rule which we use to simplify our language' " (Clifton & Monton, 1999, p. 716). Hence, the fuzzy link does not in any way occupy a prominent place in the theory, and for this reason suppressing the anomaly seems to be enough.

I don't think that this argument is satisfactory. Though there is nothing inherently wrong with it, it contradicts the spirit of GRW theory. This is because GRW falls into a class of proposals which attempt to salvage a firmly realist view of QM, that is one in which things have, or at least end up having, definite properties. And this must be true not only of waves, but also of ordinary objects. Cats really *are* dead or alive; the predicates "dead" and "alive" are not merely convenient jargon we introduce to facilitate our language.[5] To make the connection between the wave function and ordinary properties a mere postulate we use to simplify our language, which as such is neither true nor false, gives the theory an antirealist thrust that is foreign to its spirit. For this reason, I think, we must be able to tell a clear and anomaly-free story about how to retrieve particle properties from wave functions if we want to continue to take the theory at all seriously—merely suppressing the anomaly is not enough.

In what follows I will try to sketch how this can be achieved. The solution I will offer is simple and straightforward. There is no counting anomaly. The alleged anomaly is based on the seemingly plausible but faulty assumption that the composition principle holds in FQM. In the following section I will introduce this principle, prove that it holds true in standard quantum mechanics (SQM) and show that and how it fails in FQM. I then argue that what lies at the heart of the anomaly is the failure to appreciate this peculiar feature of the property structure of collapse interpretations. Once this is realized, the anomaly vanishes. What remains, however, is a violation of common sense. Our everyday experience tells us that the composition principle holds true for spatial properties (an intuition which is borne out in classical mechanics as well as in SQM) and it is quite irritating to realize that this is wrong in FQM. Yet it is not the first time that everyday experience turns out to be a bad guide in quantum matters, and so we should neither be too surprised nor too worried about being forced to give up an element of our common intuition.

3. The composition principle and its failure in FQM

The composition principle posits that if every object e_i of an ensemble $E = \{e_1, \ldots, e_n\}$ has property P, then the ensemble E itself has property P as well, and vice versa. Formally, $Pe_1 \& \ldots \& Pe_n$ iff PE; or if we do not restrict ourselves to finite ensembles: $\forall x(x \in E \rightarrow Px)$ iff PE. Both sides of these biconditionals refer to properties of the ensemble E. "PE" means that the *ensemble* E *itself* has property P whereas "$Pe_1 \& \ldots \& Pe_n$" expresses the fact that *every member of* E has property P. When we call the latter property $\tilde{P}$, the composition principle simply reads: $\tilde{P}E$ iff PE. This principle holds true in many cases. If a couple of objects of temperature T are put together the resulting "composite object" still has temperature T, or if all objects are blue the ensemble is blue as well. However, this principle does not always hold true. Water is wet but water molecules are not; gases have a temperature, gas molecules do not; horses have a heart, a herd of horses does not; each musician of an orchestra plays an instrument but the orchestra as a whole does not, and so on. More sophisticated examples include well-known problems from the philosophy of the social sciences: one cannot infer from the premise that every individual is rational to the conclusion that a group of individuals is rational in the same sense; or what is good for each individual need not necessarily be good for the community.

These examples highlight that to assert $\tilde{P}E$ is prima facie not the same as to assert PE. To say that every member of an ensemble has a certain property P is different from saying that the ensemble itself has this property—$\tilde{P}$ and P are two distinct properties, and $\tilde{P}E$ and PE are not logically equivalent. As a consequence, $\tilde{P}E$ and PE cannot be used interchangeably. If we nevertheless wish to do so, the composition principle has to be invoked to "bridge the gap" between the two. This principle, however, is not a truth of logic and its validity in a given context needs to be justified. If we fail to provide such a justification and assume, without further argument, that the composition principle holds true, we are guilty of a fallacy of composition.

How does this bear on the marbles? Let e_i, $i = 1,\ldots, n$, stand for the marbles and $E = \{e_1, \ldots, e_n\}$ for the ensemble of all marbles . Now, everything that has been said so far about properties of ensembles and their members also applies to the property "being in the box" all members of the ensemble E being in the box and the ensemble E itself being in the box are two different states of affair. Despite their seeming equivalence, it is prima facie not the same to assert that all members of the ensemble E are in the box and to assert that the ensemble E itself is in the box.

One might now be inclined to dismiss this point as futile logical hair-splitting, since "being in the box", or more generally "being located within the interval R", seems to be a clear example of a property for which the composition principle holds: if all members of E are located in the interval R then the ensemble E itself is located within R as well. In this section I will prove that this intuition, though borne out in SQM, fails in FQM. The situation is the following. In SQM it is possible to prove the composition principle as relating to position as a theorem, and as a consequence spatial properties of an ensemble and spatial properties of its members can be used interchangeably—as we would expect it to be. This, however, is no longer true in FQM. Within this framework, the composition principle is provably false and therefore properties of ensembles and properties of their members must be carefully distinguished. And it is this peculiarity of FQM, I will argue, that lies at the heart of the so-called counting anomaly.

The composition principle

Consider an ensemble in a disentangled state $\psi(r_1,\ldots, r_n) = \psi_1(r_1)\ldots\psi_n(r_n)$. Furthermore, notice (for details see the Appendix) that we retrieve the usual definitions of a property in SQM if we set $\varepsilon = 0$ and replace " $\geq$ " by " $=$ " in equation (1); for this reason I drop the subscript "ε" and just write $P_{R_i}(\cdot)$ and $P_{R_1\times\ldots\times R_n}(\cdot\ ,\ldots,\ \cdot)$, respectively, where $R_1, \ldots, R_n$ are finite but otherwise arbitrary intervals. Then one can prove that the following holds in SQM.

Composition principle (CP):

$P_{R_1}(e_1)\&\ldots\&P_{R_n}(e_n)$ is true if, and only if, $P_{R_1\times\ldots\times R_n}(e_1, \ldots, e_n)$ is true.

The proof is straightforward and will be given in the Appendix. As a consequence, $P_{R_{\text{in}}}(e_1)\&\ldots\&P_{R_{\text{in}}}(e_n)$ and $P_{R_{\text{in}}\times\ldots\times R_{\text{in}}}(e_1, \ldots, e_n)$ can be used interchangeably in SQM, that is, the ensemble E is in the box if all its members are in the box as well, and vice versa—just as we intuitively expect it to be.

This situation changes drastically in FQM. A brief look at the proof of CP reveals that the implication which goes from left to right no longer holds when one moves from SQM to FQM, and therefore CP is not valid any more (see again the Appendix for details). The best we can obtain in this case is the following.

Restricted composition principle (RCP):

If $P_{\varepsilon, R_1\times\ldots\times R_n}(e_1, \ldots, e_n)$ is true, then $P_{\varepsilon, R_1}(e_1)\&\ldots\&P_{\varepsilon, R_n}(e_n)$ is true as well, but *not* vice versa.

Applied to the marbles case RCP says that if the ensemble of all marbles is in the box, then every one of its members is in the box as well. The converse, however, is false: if every member of the ensemble, i.e. every individual marble, is in the box, the same need not be true for the ensemble. Admittedly, this is counter-intuitive, but that simply is how things are in FQM.

The failure of the composition principle and the counting anomaly

I now argue that what lies at the heart of the anomaly is an unwarranted use of the composition principle. To see how this comes about note that, from a logical point of view, the anomaly amounts to holding the following three contradictory statements. (1) $P_{\varepsilon, R_{\text{in}}}(e_1)\&\ldots\&P_{\varepsilon, R_{\text{in}}}(e_n)$ and $P_{\varepsilon, R_{\text{in}}\times\ldots\times R_{\text{in}}}(e_1, \ldots, e_n)$ are the same, (2) $P_{\varepsilon, R_{\text{in}}}(e_1)\&\ldots\&P_{\varepsilon, R_{\text{in}}}(e_n)$ is true, (3) $P_{\varepsilon, R_{\text{in}}\times\ldots\times R_{\text{in}}}(e_1, \ldots, e_n)$ is false.[6] Moreover, note that the anomaly does not arise in SQM because premise (1) holds and $P_{\varepsilon, R_{\text{in}}\times\ldots\times R_{\text{in}}}(e_1, \ldots, e_n)$ is true, hence the three statements are consistent.

While there is nothing wrong with (2) and (3), (1) is false. Since CP fails in FQM there is no reason to identify the two. This changes the situation drastically. If (1) is removed from the argument no contradiction can be derived—and with the contradiction the anomaly vanishes as well.

This needs some spelling out. In order to see how driving a wedge between $P_{\varepsilon, R_{\text{in}}}\&\ldots\&P_{\varepsilon, R_{\text{in}}}$ and $P_{\varepsilon, R_{\text{in}}\times\ldots\times R_{\text{in}}}$ dissolves the anomaly, some reflection on the nature of these propositions and the properties of the system they are ascribed to is required.[7]

How do we check that all marbles are in the box? I take it that what we do is no more and no less than making sure first that marble 1 is in the box, second that marble 2 is in the box, and so on through marble n. If this is the case, then all n marbles are in the box. Lewis (1997, pp. 320–321) refers to this as the “enumeration principle”. That is, we check one marble after the other and if we find each of them in the box then all are in the box. Given this procedure, the only thing we need in order to have all n marbles neatly in the box is that $P_{\varepsilon, R_{\text{in}}}\&\ldots\&P_{\varepsilon, R_{\text{in}}}$ is true.

“But what about $P_{\varepsilon, R_{\text{in}}\times\ldots\times R_{\text{in}}}$? Doesn’t it represent the state of affairs of all marbles being in the box just as well?”, one might now ask. No it doesn’t—that is the crucial thing to realize. The procedure for ensuring that all marbles are in the box as described above does not square with this proposition. There is no reason to assume that $P_{\varepsilon, R_{\text{in}}\times\ldots\times R_{\text{in}}}$ should be true if the only thing we do is to observe one marble after the other and to make sure that it is in the box. Or to put it differently, the assumption that $P_{\varepsilon, R_{\text{in}}}$ represents the state of affairs of all marbles being in the box is unwarranted.

One might now try to resist this point by arguing that it is (at least intuitively) obvious that $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ represents the state of affairs of all marbles being in the box, regardless of whether or not it squares with the above procedure. But this reply is effectively undercut by the failure of CP in FQM. From what has been said so far it is clear that $P_{\varepsilon,\, R_{\mathrm{in}}}\&\ldots\&P_{\varepsilon,\, R_{\mathrm{in}}}$ *does* represent the state of affairs at stake. Therefore, if we want to establish that $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ equally does, we are (at least) committed to the claim that these two propositions are true of the same things (i.e. that they are extensionally equivalent). A minimal condition for this to be correct is that the two have the same truth conditions. But this is not the case, as the failure of CP instructs us. There are cases where $P_{\varepsilon,\, R_{\mathrm{in}}}\&\ldots\&P_{\varepsilon,\, R_{\mathrm{in}}}$ is true while $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ fails. For this reason, the two are not extensionally equivalent and I conclude that, provided we grant that the former expression represents the state of affairs of all marbles being in the box, the latter fails to do so.

But if $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ does *not* represent the state of affairs of all marbles being in the box, what then does it represent? The property has so far always been paraphrased as "the ensemble being in the box". This nice phrase masks the fact that we actually don't have any firm grip on what this property is. When we think about an ensemble of marbles, what we have in mind is, roughly speaking, a bunch of individuals sitting there in the box. But this is precisely not what $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ expresses: it just is *not* the property of each marble sitting in the box. Although $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ implies that all marbles sit in the box (by RCP), this does not exhaust its meaning, as the failure of CP shows. But what then does?

I have no answer to this question; and I think we don't need one. First, the interest in $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ is based on the belief that it reflects the "counting property", but, as I have argued, this is not the case. For this reason we don't need it and we don't yet have a need to worry about its interpretation. Second, it is in general a mistake to think that everything we can define in the formalism represents something interesting in the world. Not every expression we can write down corresponds to a property that is physically relevant and that is accessible to observation. Some expressions that the formalism allows for may be no more than mathematical constructs not amenable to measurement and without physical significance; and $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ may well belong to this class.

To sum up, we don't have to bother about $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ because, first, it does not play any role in the problem at hand (since it does not represent the state of affairs of all marbles being in the box) and, second, there is no prima facie reason to assume that it represents anything of physical interest.

Once we get rid of the faulty identification of the two propositions $P_{\varepsilon,\, R_{\mathrm{in}}}\&\ldots\&P_{\varepsilon,\, R_{\mathrm{in}}}$ and $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ the contradiction, and with it the anomaly, vanishes. We put n marbles in the box and indeed end up having them there; we have been fooled into believing that they are not by the falsity of $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$. But this is just not relevant to the issue of where the marbles are. The failure of CP in FQM has the counter-intuitive consequence that we are forced to divorce two propositions which intuitively seem to be the same (or at least extensionally equivalent)—an intuition which is borne out in classical mechanics as well as in SQM. But this is a matter of fact about propositions and not an anomaly.

To drive my point home I have to deal with a further problem. There is an argument—endorsed by Lewis (1997, p. 320) and echoed in Clifton and Monton (2000, p. 160)—for the conclusion that it is unacceptable to assert that all marbles are in the box on the grounds that there is a vanishing probability in the state $|\psi_{\mathrm{total}}\rangle = (a|\psi_{\mathrm{in}}\rangle + b|\psi_{\mathrm{out}}\rangle)_1 \ldots (a|\psi_{\mathrm{in}}\rangle + b|\psi_{\mathrm{out}}\rangle)_n$ of finding them there. The argument is straightforward and runs as follows. Born's rule tells us that the probability of finding the system in state

$|\psi_{in}\rangle_1 \ldots |\psi_{in}\rangle_n$ is $|a|^{2n}$; and since $|a|^{2n} \leq 1$, there is a vanishing probability of finding all the marbles in the box.

This argument is flawed. But it is flawed in an interesting way because it draws our attention to an issue that does not seem to be much discussed, namely how to calculate probabilities in FQM. Given that FQM alters the conditions for a property to obtain, one would expect that the way to calculate the probability for this to happen has to be altered as well. In the remainder of this section I argue that this is indeed the case and show that the above argument is flawed because it uses a way of calculating probabilities adequate to SQM but not to FQM.

To get the gist, consider a marble in state $|\psi_m\rangle = a|\psi_{in}\rangle + b|\psi_{out}\rangle$. What is the probability p of it being true that the marble is in the box? In SQM we associate this with the eigenstate $|\psi_{in}\rangle$ and using Born's rule we get $p = |a|^2$. However, in FQM by definition it is true that a system is in the box if it is in some state $|\psi_m\rangle$ where $|a|^2 \geq 1 - \varepsilon$. Given this, it does not make sense to say that the probability of finding the marble in the box equals $|a|^2$ in $|\psi_m\rangle$. We cannot both define the conditions such that the proposition is true when the system is in state $|\psi_m\rangle$ and at the same time take the probability of the proposition to be true to be smaller than 1. This is contradictory. If we allow a proposition to be true in non-eigenstates, we have to take these same non-eigenstates when using Born's rule to calculate probabilities. In the present example, perhaps one might say that the FQM probability of a marble in state $|\psi\rangle$ to be in the box is $|\langle\psi|\psi_m\rangle|^2$, and not $|\langle\psi|\psi_{in}\rangle$, as SQM has it.[8]

From this it is clear where the rub lies. It is true that the probability of finding the system in state $|\psi_{in}\rangle_1 \ldots |\psi_{in}\rangle_n$ is vanishingly small, but from this it does *not* follow that the probability of finding all the marbles *in the box* is equally small. It is the leading idea of FQM that less than a precise eigenstate is needed in order for a property to obtain. However, Lewis' argument infers from the fact that the probability of finding the system in state $|\psi_{in}\rangle_1 \ldots |\psi_{in}\rangle_n$ is small that the probability of finding it in the box is equally small, and thus implicitly associates "being in the box" with the state $|\psi_{in}\rangle_1 \ldots |\psi_{in}\rangle_n$. Thereby it carries over to FQM a way of thinking about probabilities that is inappropriate to it. It is the whole point of FQM that it is too restrictive to require the system to be in a precise eigenstate in order for it to be true that the marbles are in the box; "being in the box" can be true in a state that is somewhat close but not equivalent to $|\psi_{in}\rangle_1 \ldots |\psi_{in}\rangle_n$. For this reason it is true in SQM but not in FQM that the probability of finding all marbles in the box equals $|a|^{2n}$. The correct probability of finding all marbles in the box seems to be $p = |\langle\psi_{total}\,|(|\psi_m\rangle_1 \ldots |\psi_m\rangle_n)|^2$, which immediately yields that the probability of finding all marbles in the box is one, as one would expect within FQM.

4. Objections

In this section I consider two objections to the dissolution of the anomaly I have just presented.

First, to discuss a property of the ensemble as a whole, one must represent that property using a "collective variable" (such as the centre of mass of the total system, for instance) and a mere n-tuple of positions does not make up a collective variable.[9] For this reason, $P_{\varepsilon, R_1 \times \ldots \times R_n}$ does *not* truly reflect a property of the ensemble but is merely another way to describe a bunch of single particle properties. But since I maintain that the anomaly is dissolved by realizing that the property of the ensemble being in the box is not equivalent to the property of each and every marble individually being in the box, my argument depends on the claim that $P_{\varepsilon, R_1 \times \ldots \times R_n}$ indeed reflects a claim about the ensemble. Hence, so the objection goes, my argument is flawed because it turns on this faulty assumption.

I don't think that this argument is conclusive. Representability by a "collective variable" is certainly a sufficient, but not a necessary condition for a collective property. The underlying intuition of this objection seems to be that $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$ does not make a genuine claim about the ensemble because the operator involved is just a tensor product, and as such merely "patches together" single marble properties without adding anything to them.

This, however, is to carry over to FQM a criterion of identity from SQM that is no longer appropriate. To individuate a property it is *not* sufficient to specify an operator; we also have to provide truth conditions. And it is at this point where SQM and FQM diverge.[10] Due to the fact that CP obtains in SQM the truth conditions for $P_{R_1 \times \ldots \times R_n}$ are equivalent to the ones for $P_{R_1}\&\ldots\&P_{R_n}$ and for this reason it is true that forming a product does not add anything to the properties possessed by the individuals.

However, this is no longer the case in FQM. The truth conditions for $P_{\varepsilon,\, R_1 \times \ldots \times R_n}$ are, as the failure of CP shows, different from the ones for $P_{\varepsilon,\, R_1}\&\ldots\&P_{\varepsilon,\, R_n}$. Therefore, I think, it reflects a genuine property of the ensemble.

The second objection slightly changes the set-up of the experiment and considers n individual boxes, one for each marble, instead of one big box. Then, so the argument goes, one cannot even commit a fallacy of composition because committing the fallacy involves the attribution of the *very same* property (being in the box) to the ensemble as a whole and to its members. But if every marble is in a different box, spatially separated from all the other boxes, there simply is no such property because there is not even a uniform property assigned to the marbles.

To meet this objection it suffices to realize that nothing in the above argument hinges on the fact that all marbles are within the *same* box. Putting all the marbles in different boxes (instead of just one box) amounts to replacing $P_{\varepsilon,\, R_{\mathrm{in}}}\&\ldots\&P_{\varepsilon,\, R_{\mathrm{in}}}$ by $P_{\varepsilon,\, R_1}\&\ldots\&P_{\varepsilon,\, R_n}$, where $R_1, \ldots, R_n$ are non-overlapping intervals associated with the n one-marble boxes, and substituting $P_{\varepsilon,\, R_1 \times \ldots \times R_n}$ for $P_{\varepsilon,\, R_{\mathrm{in}} \times \ldots \times R_{\mathrm{in}}}$. The "individual box version" of the anomaly then is: E has the property that one of its members is within R_1, one within R_2, …, and one within R_n while, as an ensemble, it fails to be located within the n-dimensional interval $R_1 \times \ldots \times R_n$. Logically, this comes to holding the following contradictory statements. (1) $P_{\varepsilon,\, R_1}\&\ldots\&P_{\varepsilon,\, R_n}$ and $P_{\varepsilon,\, R_1 \times \ldots \times R_n}$ are the same, (2) $P_{\varepsilon,\, R_1}\&\ldots\&P_{\varepsilon,\, R_n}$ is true and (3) $P_{\varepsilon,\, R_1 \times \ldots \times R_n}$ is false.

But by now it is obvious that this does not pose any threat to my line of argument. Even if all intervals R_j are different, the failure of CP, as formulated in Section 3, assures that the truth of $P_{\varepsilon,\, R_1 \times \ldots \times R_n}$ does not follow from the truth of $P_{\varepsilon,\, R_1}\&\ldots\&P_{\varepsilon,\, R_n}$. For this reason premise (1) is false. As a consequence, the anomaly vanishes, just as in the "single box version" of the argument. So, after all, it has not much bearing on the anomaly whether one thinks of all the marbles being put in one single box or of each marble being located in an individual box, spatially separated from all the others.

5. Facing the consequences

Where does this leave us? I have argued that since we are dealing with a bunch of non-interacting marbles, it is sufficient for the marbles to be in the box that $P_{\varepsilon,\, R_{\mathrm{in}}}\&\ldots\&P_{\varepsilon,\, R_{\mathrm{in}}}$ holds. Nothing else is needed. Since this conjunction holds true by assumption, each marble is neatly in the box as we expect it to be and no anomaly pops up. And similarly for arithmetic. Since counting is a process that is concerned with individual objects rather than with ensembles as a whole, the thing we need in order to count is that the conjunction of all $P_{\varepsilon, R_{\mathrm{in}}}(e_j)$ is true. The general lesson we learn from this discussion is that the property structure of FQM is far more complex than that of SQM—that is the price we have to pay for the admission of non-eigenstates as property-bearing states.

All this may strike one as rather peculiar and one might be inclined to interpret the failure of CP and the resulting proliferation of properties as a *reductio ad absurdum* of FQM. Although one certainly can (and many probably will) adopt this point of view, it is by no means compelling to do so. The failure of compositionality, in some form or another, is a problem that besets other interpretations of QM (some brands of the modal interpretation, see Clifton, 1996, pp. 385ff.) and more generally other domains of philosophy as well. In the remainder of this section I briefly discuss how the problem arises in epistemology and draw some parallels to the failure of CP in FQM.

Consider a lottery of a million tickets and just one price. Hence I have good reasons to believe that the one ticket I bought will not win. But the same argument goes through for every ticket, and it is therefore rational to believe that each ticket will not win. Thus, I seem justified in believing that ticket No. 1 will not win, and ticket No. 2 will not win, and …, ticket No. 1,000,000 will not win. If we now assume that the (informal version of the) composition principle (in this context often referred to as the "conjunction principle") holds—i.e. that given we are justified in believing p, and we are also justified in believing in q, then we are justified in believing $(p\&q)$—then we come to the conclusion that no ticket at all will win. This is the by now well-known *lottery paradox*, first developed in Kyburg (1961, p. 197). The conclusion is false but apparently justified. So we are in the awkward position of being justified to believe in the conjunction of individually justified propositions although we know that it is false. This paradox gave rise to extended debates and there is no generally agreed-upon solution, but one obvious way to evade the difficulty is to reject the composition principle for justified belief.

Another related epistemic paradox arises if we posit (plausibly) that we know p if our subjective probability that p is true is at least 0.95. However, adopting the composition principle we run into the same problem. Given we know $p_1, \ldots, p_{100}$ with probability 0.95, the probability of $p_1\&\ldots\&p_{100}$ is 0.95^{100} which is much smaller than 0.95. Hence, we don't know $p_1\&\ldots\&p_{100}$ although we know $p_1, \ldots, p_{100}$ individually.

There are striking similarities between these epistemic paradoxes and the counting anomaly. Both deal with individual objects that have some property while a collection of individuals, which we intuitively would expect to have the same property, actually fails to do so. And in all cases there are good reasons to lay the blame on the composition principle. Nevertheless we keep taking concepts like knowledge and belief seriously and take these paradoxes to be a challenge for future research rather that a reason to give up on the issue all together. Why not adopt the same attitude towards properties in FQM? CP fails and properties proliferate, but that can also be taken as setting the agenda for further investigation and need not lead to the damnation of the theory.

Appendix: proof of the composition principle in SQM

First, note that we obtain the usual definition of a property in SQM when the integrals on the left-hand side of equation (1) are set equal to one. In some more detail, the argument runs as follows. A system in state $|\psi\rangle$ has the property U iff $|\langle\psi \mid e_u\rangle|^2 = 1$, where

$|e_u\rangle$ is the state in the Hilbert space associated with the property U. This is equivalent to the condition $\langle\psi|\hat{P}_{e_u}|\psi\rangle = 1$, where $\hat{P}_{e_u}$ is the projection operator on $|e_u\rangle$. If there is not just one single vector, but an entire subspace S_u of the Hilbert space associated with U, the condition reads $\langle\psi|\hat{P}_{S_u}|\psi\rangle = 1$, where $\hat{P}_{S_u}$ is the projection operator on the subspace S_u. Now choose U to be "being located within interval $R_1 \times \ldots \times R_n$". Then this condition reads $\langle\psi|\hat{P}_{R_1\times\ldots\times R_n}|\psi\rangle = 1$. Now expand both $|\psi\rangle$ and $\hat{P}_{R_1\times\ldots\times R_n}$ in the position basis: $|\psi\rangle = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} \mathrm{d}^n r \psi(r_1, \ldots, r_n)|r_1\ldots r_n\rangle$ and $\hat{P}_{R_1\times\ldots\times R_n} = \int_{R_1}\ldots\int_{R_n} \mathrm{d}^n r |r_1\ldots r_n\rangle\langle r_1\ldots r_n|$. Plugging this into the above condition (after some calculations) yields: $\langle\psi|\hat{P}_{R_1\times\ldots\times R_n}|\psi\rangle = \int_{R_1\times\ldots\times R_n}|\psi(r_1, \ldots, r_n)|^2\mathrm{d}^n r = 1$, which is equation (1) with the aforementioned changes.

This said, we are now in a position to prove that CP holds for properties thus defined.

$\Rightarrow$: Assume $P_{R_1}(e_1)\&\ldots\&P_{R_n}(e_n)$ holds, that is, $\int_R |\psi_i(r_i)|^2 \mathrm{d}r_i = 1$; $i = 1, \ldots, n$. Since we built up our collective "n-marble entity" from n non-interacting marbles the state will not be entangled and can be written as the product of the states of the individual marbles: $\psi(r_1, \ldots, r_n) = \psi_1(r_1)\ldots\psi_n(r_n)$; and since in SQM the wave functions of a well-behaved quantum state is integrable we can factorize the integral: $\int_{R_1\times\ldots\times R_n}|\psi_1(r_1)\ldots\psi_n(r_n)|^2 \mathrm{d}^n r = \int_{R_1}|\psi_1(r_1)|^2 \mathrm{d}r_1\ldots\int_{R_n}|\psi_n(r_n)|^2 \mathrm{d}r_n$. By assumption all terms of this product equal one, hence $\int_{R_1\times\ldots\times R_n}|\psi_1(r_1)\ldots\psi_n(r_n)|^2\mathrm{d}^n r = 1$. QED.

$\Leftarrow$: Assume $P_{R_1\times\ldots\times R_n}(e_1, \ldots, e_n)$ holds, that is, $\int_{R_1\times\ldots\times R_n}|\psi_1(r_1)\ldots\psi_n(r_n)|^2\mathrm{d}^n r = 1$. Factorize the integral as above: $\int_{R_1\times\ldots\times R_n}|\psi_1(r_1)\ldots\psi_n(r_n)|^2\mathrm{d}^n r = \int_{R_1}|\psi_1(r_1)|^2\mathrm{d}r_1\ldots\int_{R_n}|\psi_n(r_n)|^2 \mathrm{d}r_n = 1$. It is an axiom of SQM that $\int_{R_i}|\psi_i(r_i)|^2\mathrm{d}r_i \leq 1$ for all $i = 1, \ldots, n$. For this reason the above product can equal 1 only if $\int_{R_i}|\psi_i(r_i)|^2\mathrm{d}r_i = 1$ for all $i = 1, \ldots, n$. QED. This completes the proof of CP for SQM.

Furthermore, it is straightforward to see that the first half of the proof no longer goes through if the SQM definition of a property is replaced by the fuzzy link; the second part, however, is not affected by this change. For this reason, CP does not hold in FQM, but RCP does.

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Notes

1. A classical source for this rule is Dirac (1930, pp. 46–47).
2. The choice of this term is somehow arbitrary, one might just as well use "system", "collective", or "composite entity".
3. I should note that FQM does not exhaust *all* collapse interpretations of QM since there may be methods other than the fuzzy link to associate properties with non-eigenstates. In particular, there is the so-called mass–density interpretation now favoured by Ghirardi and co-workers (Ghirardi *et al.*, 1995; Bassi & Ghirardi, 1999). Space constraints prevent me from discussing this approach here. However, not much seems to be lost by this omission since, as Clifton and Monton (2000, pp. 156–161) point out, the anomaly equally arises under this interpretation. Moreover, their discussion shows that it arises in the

same way and for the same reasons as under the fuzzy link interpretation. For this reason, my arguments in what follows carry over *mutatis mutandis* to an interpretation of QM based on the mass–density approach.

4. This, of course, may well involve an unrealistically large number of marbles. As Ghirardi and Bassi (1999, p. 55) have pointed out, more than the entire mass of the universe may be needed to produce the required number of marbles. But this does not matter in the present context. Although such considerations my be important for practical matters, they have no force when it comes to foundational issues.
5. This seems also to be the view of Ghirardi and co-workers. They never denied that a tidy connection between waves and "ordinary" properties must be established (Ghirardi *et al.*, 1995).
6. The characterization of the anomaly in logical terms is in line with Clifton and Monton (1999, pp. 700 and 703). However, Lewis' emphasis is on the violation of common sense and not on logical structure. But this difference is one of style rather than of substance, since what does violence to common sense is the denial of premise (1) which is implicitly endorsed.
7. To facilitate notation I drop the brackets in what follows and write $P_{\varepsilon,\, R_{\mathrm{in}}}\&\ldots\&P_{\varepsilon,\, R_{\mathrm{in}}}$ instead of $P_{\varepsilon,}$ $R_{\mathrm{in}}(e_1)\&\ldots\&P_{\varepsilon,\, R_{\mathrm{in}}}(e_n)$ and $P_{\varepsilon,\, R_{\mathrm{in}}\times\ldots\times R_{\mathrm{in}}}$ instead of $P_{\varepsilon,\, R_{\mathrm{in}}}\times\ldots\times R_{\mathrm{in}}(e_1, \ldots, e_1)$.
8. A problem with this suggestion is that the choice of $|\psi_{\mathrm{m}}\rangle$ is ambiguous. Since according to the fuzzy link, all $|\psi_{\mathrm{m}}\rangle$ with $|a|^2 \geq 1-\varepsilon$ have property at stake, any will do. One possible solution to this problem is to choose the state with the smallest admissible a ($|a|^2 = 1-\varepsilon$) and stipulate that p equals $|\langle\psi|\psi_{\mathrm{m}}\rangle|^2$ for all states whose coefficient of $|\psi_{\mathrm{in}}\rangle$ is smaller than a and 1 for all states with this coefficient greater that a. The last clause is needed to prevent that a state which is closer to the eigenstate $|\psi_{\mathrm{in}}\rangle$ than $|\psi_{\mathrm{m}}\rangle$ is assigned a probability smaller than one of being in the box. This is a workable suggestion, but it admittedly has the air of ad hocness to it; the issue of how to calculate probabilities in FQM will certainly need further consideration.
9. I am grateful to Rob Clifton for having drawn my attention to this point.
10. Thanks to Nancy Cartwright for having pointed this out to me.

References

ALBERT, D.Z. & LOEWER, B. (1995) Tails of Schrödinger's cat, in: R. CLIFTON (Ed.) *Perspectives on Quantum Reality: Non-relativistic, relativistic, and Field-theoretic* (Dordrecht, Kluwer Academic, pp. 81–92.

BASSI, A. & GHIRARDI, G. (1999) More about dynamical reduction and the enumeration principle, *British Journal for the Philosophy of Science*, 50, pp. 719–734.

BASSI, A. & GHIRARDI, G. (2001) Counting marbles: reply to Clifton and Monton, *British Journal for the Philosophy of Science*, 52, pp. 125–130.

BELL, J.S. (1987) Are there quantum jumps?, in: J.S. BELL, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge, Cambridge University Press), pp. 201–212.

CLIFTON, R. (1996) The properties of modal interpretations of quantum mechanics, *British Journal for the Philosophy of Science*, 47, pp. 371–398.

CLIFTON, R. & MONTON, B. (1999) Losing your marbles in wavefunction collapse theories, *British Journal for the Philosophy of Science*, 50, pp. 697–717.

CLIFTON, R. & MONTON, B. (2000) Counting marbles with "accessible" mass density: a reply to Bassi and Ghirardi, *British Journal for the Philosophy of Science*, 51, pp. 155–164.

DIRAC, P.A.M. (1930) *The Principles of Quantum Mechanics* (Oxford, Oxford University Press).

GHIRARDI, G. & BASSI, A. (1999) Do dynamical reduction models imply that arithmetic does not apply to ordinary macroscopic objects?, *British Journal for the Philosophy of Science*, 50, pp. 49–64.

GHIRARDI, G., GRASSI, R. & BENATTI, F. (1995) Describing the macroscopic world: closing the circle within the dynamic reduction program, *Foundations of Physics*, 25, pp. 5–38.

GHIRARDI, G., RIMINI, A. & WEBER, T. (1986) Unified dynamics for microscopic and macroscopic systems, *Physical Review*, 34D, pp. 470–491.

KYBURG, H. (1961) *Probability and the Logic of Rational Belief* (Middletown, CT, Wesleyan University Press).

LEWIS, P.J. (1997) Quantum mechanics, orthogonality, and counting, *British Journal for the Philosophy of Science*, 48, pp. 313–328.

NEUMANN, J. VON (1955) *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ, Princeton University Press 1955).

SAINSBURY, R.M. (1995) *Paradoxes*, 2nd edn (Cambridge, Cambridge University Press).

SAKURAI, J.J. (1994) *Modern Quantum Mechanics* (Reading, MA, Addison-Wesley).

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