A New Theory of Content II: Model Theory and Some Alternatives

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1. Review of the Syntactical Specification of Content

Gemes (1994) offered two equivalent syntactic definitions of so-called basic (logical) content for a generic propositional language $L_{\sim v\&\rightarrow}$, hereafter L. The prime motivation for the theory of content was to construct a notion of consequence that would not allow arbitrary disjunctions to count as part of the contents of their disjuncts; for instance, to disallow '(pvq)' from counting as part of the content of 'p'. Gemes (1994) was framed in terms of "basic content" rather than simply "content" because otherwise it would have yielded an account according to which disjunctions of content parts did not count as content parts. I have come to consider this result acceptable, perhaps even desirable.[1] So for the remainder of this paper I will treat Gemes (1994) as if it gave an account of content rather than basic content.

Let 'α', 'ß' and 'ø' be variables for wffs, and, were specified, sets of wffs, of the propositional language L; '$\vdash$' stand for the classically defined relation of being a syntactical consequence; and α_{dnf} be a canonical Boolean disjunctive normal form of arbitrary (contingent) L wff α in the essential vocabulary of α. For instance, presuming 'p', 'q' and 'r' are atomic wffs of L, $((pvq)\&(rv{\sim}r))_{dnf}$ is '(p&q)v(p&~q)v(~p&q)'.[2] Then the following, using '$ß \vdash_c \alpha$' to abbreviate 'α is a (syntactically specified) content part of ß' and '$ß \nvdash_c \alpha$' to abbreviate 'α is not a (syntactically specified) content part of ß', is a variant of one of Gemes (1994)'s definitions of content for wffs of L

(D1) $ß \vdash_c \alpha =_{df}$ (i) α and ß are contingent, (ii) $ß \vdash \alpha$, and (iii) each disjunct of α_{dnf} is a sub-conjunction of some disjunct of $ß_{dnf}$.[3]

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[1] A referee from this journal pointed out that since the content relationship is closed under logical equivalence if one counts 'pvq' as part of the content of 'p&q' then one must equally, and contra many people's natural intuitions, count '~p→q' as part of the content of 'p&q'. The same referee also helpfully pointed me in the direction of Stelzner (1992).

[2] For a more precise definition see Gemes (1994). Note,canonical disjunctive normal forms contain no redundant, that is logically equivalent, disjuncts. The notion of dnf remains undefined for non-contingent wffs.

[3] Gemes (1994) actually gave two syntactic definitions of content for L, (BCPL1) and (BCPL2) of Gemes (1994), and proved they were extensionally equivalent. (D1) here is similar to (BCPL2) of Gemes (1994) which, replacing Gemes' (1994) '$\alpha <_b ß$' notation with our current '$ß \vdash_c \alpha$' notation, reads

$ß \vdash_c \alpha =_{df}$ (i) α and ß are contingent, (ii) $ß \vdash \alpha$, and (iii) there is no μ such that μ is a proper sub-disjunction of α_{dnf} and $ß \vdash \mu$.

A proof of the equivalence of this definition and our current (D1) is presented in an Appendix below. The (D1) definition is used here because it lends itself to a ready transformation to a model-theoretic analog and because it, unlike (BCPL1) and (BCPL2), allows for a facile proof of transitivity.

In (D1), while 'α' ranges over single wffs, the variable 'ß' may be interpreted as ranging over both single wffs and sets of wffs. So (D1) may be taken as given both the contents of wffs and the content of theories, where theories are taken to be sets of wffs. From here on we shall generally exclude mention of the notion of (D1) and subsequent definitions giving the content of theories and concentrate on the notion of the content of wffs. For more on the content of theories see Gemes (1993).

Some examples of content relationships for wffs of L:

$p \vdash_C p$	$p \nvdash_C (pvq)$
$p \vdash_C (p\&(rv{\sim}r))$	$p\&(pvq) \nvdash_C (pvq)$
$(p\&(rv{\sim}r)) \vdash_C p$	$(p\&q) \nvdash_C (pvq)$
$(p\&q) \vdash_C p$	$(pq) \nvdash_C (p\rightarrow q)$
$(p\&q) \vdash_C q$	$(p\&q) \nvdash (pq)$
$((p\&q)v(r\&s)) \vdash_C (pvr)$	$p\&(qvr) \nvdash_C qv(p\&r)$
${\sim}p\&(pvq) \vdash_C q$	$q \nvdash_C (p\rightarrow q)$

As noted in Gemes (1994) this type of definition amounts to saying that α is a content part of ß, iff α and ß are contingent and α is the strongest consequence (up to logical equivalence) of ß constructible using just the essential atomic wffs occurring in α.[4] Furthermore such definitions readily admit of a mechanical decision procedure [Cf. Gemes (1994), p.606 and pp.610-611].

2. A Model-Theoretic Definition of Content for L.

We may easily create a model-theoretic analog of (D1). Such accounts are read of from (D1) since it makes use of disjunctive normal forms and such forms have obvious model-theoretic analogs.

Each disjunct of the dnf form of a wff α may be seen as corresponding to an "α-relevantly specified model of α", or, more simply, an "α-relevant model of

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4 Indeed, this strongest-consequence formulation is a paraphrase of the definition BCPL1 given in Gemes (1994). Later in section 7. we shall see that this restriction to consequences constructible in the essential vocabulary of α makes our definition of content less suitable than it might be for the purpose of giving a formal account of Gricean conversational maxims. If we dropped the reference to essential vocabulary we would have a notion of content which would not guarantee that logical equivalents wffs have the same content relationship to other wffs. For instance, such a definition would have 'p' but not 'p&(qv~q)' be part of the content of 'p&q'. Such a failure to treat logical equivalents equivalently would wreck havoc when applied to problems in the philosophy of science, for instance, in defining confirmation relations. However for other purposes it would be a boon.

α". For arbitrary L wff α, an α-relevant model of α is a model of α that specifies values for all and only those propositional variables whose truth values on some model or other are relevant to the truth value of α.

For instance where α is '(pvq)', α_{dnf} is '(p&q)v(p&˜q)&(˜p&q)'. Here the first disjunct corresponds to that α-relevant model of α which assigns 'p' the value true and 'q' the value true and makes no other assignments (to propositional variables). The second disjunct corresponds to that α-relevant model of α that assigns 'p' the value true and 'q' the value false and makes no other assignments. The third disjunct corresponds to the α-relevant model of α where 'p' is assigned the value false and 'q' true and makes no other assignments. Now suppose ß is '(pvq)&r'. Then $ß_{dnf}$ is '(p&q&r)v(p&˜q&r)v(˜p&q&r)'. The first disjunct of $ß_{dnf}$ corresponds to the ß-relevant model of ß which assigns 'p', 'q' and 'r' the value true and makes no other assignments. The second disjunct of $ß_{dnf}$ corresponds to the ß-relevant model of ß which assigns 'p' and 'r' the value true, and 'q' the value false and makes no other assignments. The third disjunct of $ß_{dnf}$ corresponds to the ß-relevant model of ß which assigns 'p' the value false and 'q' and 'r' the value true makes no other assignments. Thus we have the following table summarizing the various relevant models for our designated values for α and ß:

α-relevant model of α	ß-relevant models of ß
1. p:T q:T	1. p:T q:T r:T
2. p:T q:F	2. p:T q:F r:T
3. p:F q:T	3. p:F q:T r:T

Now given our specified values of α and ß, α is a content part of ß according to (D1). According to (D1), where α is a content part of ß each disjunct of α_{dnf} is a sub-conjunction of some disjunct of $ß_{dnf}$. In other words, and as seen in the above table, where α is a content part of ß each α-relevant model of α is a sub-model of some ß-relevant model of ß, or, equivalently, each α-relevant model of α can be expanded to a ß-relevant model of ß.

Here already we have the makings of a model theoretic account of content. To make it more precise we need to give an exact account of what, for any wff α an α-relevant model of α is. To do this we need to introduce the notion of a partial interpretation and some other notions.

A partial interpretation of L is a valuation function which assigns to at least one propositional variable of L one of the two truth values true (T) and false (F) and assigns to no propositional variable of L both T and F.

We can take a full interpretation of L to be a partial interpretation that assigns a truth value to every propositional variable of L. Truth under a full interpretation is defined in the usual way.[5] Where wff α is true under full interpretation P we say P is a model of α.

A propositional variable ß is relevant to wff α iff there is some model P of α such that there is some interpretation P' which differs from P in and only in the value P' assigns to ß and P' is not a model of α.

An (full or partial) interpretation P' is an extension of partial interpretation P iff for any propositional variable ß, if P assigns T(F) to ß then so does P'. Where interpretation P' is an extension of interpretation P we may say that P is a sub-model of P'.

(D2.1) P is an α-relevant model of wff α of L$=_{df}$ P is a partial interpretation of L such that P assigns values to each of and only those propositional variables relevant to α and is such that for any full interpretation P' which is an extension of P, P' is a model of α.

Now, assuming the classical relation of semantical consequence ($\models$) and using 'ß $\models_C \alpha$' to abbreviate 'α is a (semantically specified) content part of ß' and 'ß $\not\models_C \alpha$' to abbreviate 'α is a not a (semantically specified) content part of ß', we are at last in a position to give a model theoretic definition of content part for wffs of L:

(D2) ß $\models_C \alpha =_{df}$ α and ß are contingent, ß $\models \alpha$, and every α-relevant model of α has an extension which is a ß-relevant model of ß.

Alternatively, we might say that ß $\models_C \alpha$, where α and ß are contingent, ß $\models \alpha$, and every α-relevant model of α is a sub-model of some ß-relevant model of ß.

According to (D2), for instance, '(pvq)' is not a content part of 'p', since that '(pvq)'-relevant model of '(pvq)' which assigns 'p' the value F and 'q' the value T and makes no other assignments has no extension which is a 'p'-relevant model of 'p'.

According to (D2), for instance, 'p' is part of the content of '(p&q)'. There is only one 'p'-relevant model for 'p', namely that which assigns T to 'p' and makes no other assignments and that model has an extension to a '(p&q)'-relevant model of '(p&q)', namely that which assigns 'p' the value T and 'q' the value T and makes no other assignments.

More generally, (D2) yields the same results as (D1) since we have the correctness and completeness theorem:

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[5] We give no definition of truth under a partial interpretation since no use is made of such a notion here.

Theorem 1: For any wffs α and ß of L, ß $\vdash_C \alpha$ iff ß $\models_C \alpha$.

Proof: Cf. definitions (D1) and (D2) and note that (i) each α-relatively specified model of α uniquely characterizes and is uniquely characterized by a unique disjunct of α_{dnf}, (ii) each ß-relatively specified model of ß uniquely characterizes and is uniquely characterized by a unique disjunct of ß$_{dnf}$ and (iii) an α-relatively specified model m of α has an extension which is a ß-relatively specified model m' of ß iff the disjunct of α_{dnf} which characterizes m is a sub-conjunction of the disjunct of ß$_{dnf}$ that characterizes m'.

The classical consequence relation is both reflexive and transitive. The content relation defined by (D2) and (D1) is clearly reflexive for any contingent wff α, thus we have

Theorem 2: For any contingent wff α of L, $\alpha \models_C \alpha$.

Proof: Trivial.

Gemes (1994) made fairly heavy going of a proof of transitivity. (D2) makes for a transparent proof of transitivity which applies equally for definitions for more sophisticated languages than L.

Theorem 3: For any wffs α, ß and ø of L, if ß $\models_C \alpha$ and $\alpha \models_C$ ø then ß $\models_C$ ø

Proof: Assume ß $\models_C \alpha$ and $\alpha \models_C$ ø. Then ß, α, and ø are all contingent; ß $\models \alpha$ and $\alpha \models$ ø; every ø-relevant model of ø is a sub-model of some α-relevant model of α; and every α-relevant model of α is a sub-model of some a ß-relevant model of ß. So, ß $\models$ ø. Further, since the relation of being a sub-model is transitive, every ø-relevant model of ø is a sub-model of some ß-relevant model of ß. So ß $\models_C$ ø.

The content relationship allows for substitution of classical equivalents. In other words

Theorem 4: For any wffs α, ß, σ, and φ of L, where $\alpha \dashv\models \beta$ and $\sigma \dashv\models \varphi$, ß $\models_C \varphi$ iff $\alpha \models_C \sigma$.

Proof: This follows from the fact that any two classically equivalent wffs α and ß share the same relevant propositional variables and hence, being equivalent, a partial interpretation P is an α-relevant model of α iff P is a ß-relevant model of ß.

3. Content for a Generic Quantification Language Without Identity

In Gemes (1994) a syntactically based definition of content for wffs of a generic quantificational language without identity was briefly considered.

Consider the generic classical quantificational language L'. The vocabulary of L' is limited to an infinite stock of individual constants, 'a', 'a_1', 'a_2', ... ; an infinite stock of individual variables 'x', 'x_1', 'x_2, ... ; an infinite stock of predicate letters of varying degrees (one-placed, two-paced, etc.), 'F', 'G', 'H', 'F_1, G_1, H_1, ... ; the sentential connectives '~', '&', 'v', '→' and '≡'; and the grouping indicators '(' and ')'. The wffs of L' are formed from the elements of L's vocabulary in the usual ways. The notions of derivable consequence ($\vdash$), semantical consequence ($\models$), contradiction, tautology, atomic wffs, basic wffs, free variables and closed wffs are defined as usual. Hereafter, unless directly specified, we shall use the term 'wffs' to refer to closed wffs. We shall use the Greek letters 'α', 'ß', and 'ø' and subscripted variants as meta-variables ranging alternatively over open wffs, closed wffs, quantifiers and individual constants of L', relying on specific indications and/or context to indicate the range in particular cases.

To define content for wffs of L we make use of the purely propositional fragment of L and the resources of propositional infinitary logics applied to that fragment of L. In particular, we allow for wffs of infinite length.

To form Dev(α) for arbitrary L' wff α (1) Put α into prenex normal form, (2) Where $ø_1.....ø_n$ A is the resultant wff, with $ø_1.....ø_n$ being a string of n, n≥0, quantifiers and A being a quantifier free wff, where n>0 eliminate the left-most quantifier $ø_n$, and (i) if $ø_n$ is a universal quantifier replace A with an infinite conjunction such that each conjunct is the result of replacing in A all occurrences of the variable governed by $ø_n$ by a given constant of L' and such that for each constant of L' there is one and only one such conjunct, or (ii) if $ø_n$ is an existential quantifier replace A with an infinite disjunction such that each disjunct is the result of replacing in A all occurrences of the variable governed by $ø_n$ by a given constant of L' and such that for each constant of L' there is one and only one such disjunct, (3) keep repeating step (2) until a quantifier free wff, being Dev(α), results.

Let α_{dnf}, that is the canonical disjunctive normal form of α, be the result of putting Dev(α) into a canonical Boolean disjunctive normal form in the essential vocabulary of Dev(α).

Then our old definition

(D1) ß $\vdash_c \alpha =_{df}$ (i) α and ß are contingent, (ii) ß $\vdash \alpha$, and (iii) each disjunct of α_{dnf} is a sub-conjunction of some disjunct of $ß_{dnf}$,

serves to define content for wffs of L'. (D1) applied to wffs of L' yields the result that 'Fa_1' is a content part of '(x)Fx'. The dnf of 'Fa_1' is 'Fa_1' and the dnf of '(x)Fx' is 'Fa_1&Fa_2& ..... & Fa_n &.... &' and clearly each disjunct of the former - we take

'Fa_1' to be a disjunction with a single disjunct - is a sub-conjunction of the

Some examples of content relationships for wffs of L':

$$(x)Fx \vdash_c Fa_1 \qquad (x)Fx \nvdash_c (x)(FxvGx)$$
$$(x)(Fx\&Gx) \vdash_c (x)Fx \qquad (x)Fx \nvdash_c (\exists x)Fx$$
$$(x)(Fx\&Gx) \vdash_c (Fa\&Gb) \qquad Fa \nvdash_c (\exists x)Fx$$

Since the (D1) definition of content applied to wffs of the quantificational language L' makes use of the classical notion of consequence applied to quantificational wffs it is not generally decidable. However the content relation, like the classical consequence relation, is decidable for the purely monadic fragment of L'.

Where α and ß are monadic wffs of L' we can use standard procedures to check whether α and ß are contingent and whether ß $\vdash \alpha$. If one of α or ß is not contingent or ß $\nvdash \alpha$ then α is not a content part of ß. Otherwise, to determine whether α is a content part of ß (1) form L'- from L' by deleting from L' all individual constants save those occurring in α or ß and then adding any two extra constants appearing in neither α or ß; then (2) form Dev(ß/L'-) and Dev(α/L'-) according to the recipe given above for forming Dev(ß) and Dev(α) substituting in the recipe all reference to L' with reference to L'-; then (3) form $\alpha_{dnf:L'-}$ and ß$_{dnf:L'-}$ by putting Dev(ß/L-) and Dev(α/L-), respectively, into a canonical Boolean disjunctive form in their essential vocabulary; then (4) check whether each disjunct of $\alpha_{dnf:L'-}$ is a sub-conjunction of some disjunct of ß$_{dnf:L'-}$. If so then α is a content part of ß otherwise it is not. Note, since both $\alpha_{dnf:L'-}$ and ß$_{dnf:L'-}$ are both finite disjunctions of finite conjunctions of atomic wffs and negated atomics wffs step (4) is mechanically decidable.

This decision procedure for the monadic case suggests a method for defining content for all wffs of L' which does not make recourse to the use of infinitary logics. To see this imagine that we restrict α_{dnf} and ß$_{dnf}$ as defined above to finite subsets of atomic wffs occurring in α_{dnf} and ß$_{dnf}$. More particularly, where ø is any arbitrary wff of L' and σ is any non-empty set of atomic wffs of L' including at least one atomic wff occurring in ø$_{dnf}$, let ø$_{dnf:\sigma}$ be ø$_{dnf}$ less all conjuncts containing atomic wffs not occurring in σ. For instance $(x)\text{~}Fx_{dnf:\{Fa,Fb\}}$ is '~Fa&~Fb'. Then we might define content for wffs of L' as follows

(D1.1) ß $\vdash_c \alpha =_{df}$ (i) α and ß are contingent, (ii) ßα, and (iii) for any finite set σ of atomic wffs of L' including at least one atomic wff occurring in α_{dnf}, each disjunct of $\alpha_{dnf:\sigma}$ is a sub-conjunction of some disjunct of ß$_{dnf:\sigma}$.

Since σ of (D1.1) is restricted to finite sets of atomic wffs in every case $\alpha_{dnf:\sigma}$

and $ß_{dnf:\sigma}$ are wffs of finite length.

(D1.1) applied to wffs of L' is equivalent to (D1) applied to wffs of L'.

4. <u>A Model-Theoretic Definition of Content for L'.</u>

We may with a bit of work create a model-theoretic analog of (D1) for wffs of L'. To do this we need to construe the quantifiers of L' substitutionally and construct our model theory accordingly. In particular, an interpretation or model for L' will be an assignment to each of the (closed) atomic wffs of L' of one and only one of the truth values T and F. A universally quantified wff α will count as true in a given model of L' iff each substitution instance of α counts as true in that model, and an existentially quantified wff will count as true in a given model of L' iff some substitution instance of it counts as true in that model.

On such a reading each disjunct of α_{dnf} for arbitrary L' wff α will correspond an "α-relevantly specified model of α", that is, a model of α that specifies values for all and only those atomic wffs whose truth values on some model or other are relevant to the truth value of α

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Let us try to be a little more precise.

A partial interpretation of L is a valuation function which assigns to at least one atomic wff of L' one of the two truth values true (T) and false (F) and assigns to no atomic wff L both T and F.

We take a full interpretation of L to be a partial interpretation that assigns a truth value to every atomic wff of L'. Truth under a full interpretation is defined in the usual way with the appropriate substitution type clauses for quantificational wffs. Where wff α is true under full interpretation P we say P is a model of α.

A (full or partial) interpretation P' is an extension of partial interpretation P iff for any atomic wff ß, if P assigns T(F)to ß then so does P'. Where P' is an extension of P and P is not an extension of P' we say P is a proper sub-interpretation of P'.

An atomic wff ß is relevant to wff α iff for some partial interpretation P, P assigns a value to ß and for every full interpretation P', if P' is an extension of P then P' is a model of α, and there is no proper sub-interpretation P'' of P, such that for every full interpretation of P''' that is an extension of P'', P''' is a model of α.

(D2.2) P is an α-relevant model of wff α of L' $=_{df}$ P is a partial interpretation of L' such that P is assigns values to each of and only those atomic wffs relevant to α and is such that for any full interpretation P' which

is an extension of P, P' is a model of α.

Given these preliminaries our old definition

(D2) ß $\models_C \alpha =_{df}$ α and ß are contingent, ß $\models$, and every α-relevant model of α has an extension which is a ß-relevant model of ß,

serves for wffs of L'.

According to (D2), for instance, 'Fa' is a content part of '(x)Fx'. The only 'Fa'-relevant model of 'Fa' is that which assigns 'Fa' the value T and makes no other assignments. By adding only the assignment T to every other wff of the form 'F' that model may be extended to an '(x)Fx'-relevant model of '(x)Fx'.

The proof of Theorem 1 above counts equally as a correctness and completeness proof for definitions (D1) and (D2) applied to wffs of L'. Thus we have

Theorem 5: For any wffs α and ß of L', ß $\vdash_C \alpha$ iff ß $\models_C \alpha$.

Similarly, the reflexivity, transitivity and substitution proofs of theorems 2, 3 and 4 work for definition (D2) applied to wffs of L'. Thus we have

Theorem 6: For any contingent wff α of L', $\alpha \models_C \alpha$.

Theorem 7: For any wffs α and ß of L' if ß $\models_C \alpha$ and $\alpha \models_C$ ø then ß $\models_C$ ø

Theorem 8: For any wffs α, ß, σ and ø of L', where $\alpha \models \sigma$ and ß $\models$ ø, ß $\models_C \alpha$ iff ø $\models_C \sigma$.

Note, while in constructing our model theoretic definition of content part for wffs of L' we have construed the quantifiers of L' substitutionally, we may otherwise give them a fully objectual reading. That is to say, one can read, for instance, '(x)Fx' as stating that every object in the domain of quantification has property F, while using recourse to a substitutional reading in order to determine the content parts of '(x)Fx'. While this dual treatment may seem infelicitous I believe that a model theoretic definition of content which eschews the use of the substitutional interpretation can be constructed along the following lines suggested by Philip Kremer.

Let M be any standard objectual denumerable model of arbitrary wff α of L'. Where D is the domain of M and E is any finite sub-domain of M we construct α(E) as follows: (1) form L'+ from L' by supplementing L' with a set I of new individual constants containing one new constant for each member of D not assigned by M to some individual constant of L', (2) form M+ from M by adding to M an assignment of each element of D not assigned by M to an individual constant of L' to a unique individual constant of I, (3) construct Dev(α) by replacing α with its non-quantificational equivalent in L+ using the recipe given in Section 3 above, substituting reference to L'+ for all references to L', (4) form Dev(α (E)) from Dev(α) by removing from Dev(α) each atomic wff containing a constant which M+ assigns to an individual that is not a member of E - where every atomic wff occurring in Dev(α) contains some constant assigned by M+ to some individual that is not a member of E, Dev(α (E)) is {/}, then (5) construct α (E) by putting Dev(α (E)) in a canonical Boolean disjunctive normal form in the essential vocabulary of Dev(α (E)) - where Dev(α (E)) is {/} then α (E) is {/}.

We might say that where M is a model of wff α with domain D, and E is a sub-domain of D, α (E) tells us in a propositional form all the truth wholly about E contained in α. We then have the preliminary definition:

(D2.2) ß $\models_C \alpha$ relative to model M of α and ß $=_{df}$ α and ß are contingent, ß $\models \alpha$, and for every finite sub-domain E of M's domain D, where α (E) is not {/}, ß(E) $\vdash_C \alpha$ (E) by (D2).

Note, ß(E) and α (E) belong to the purely propositional fragment of L'+ and hence, besides logical operators and grouping indicators, contain only individual constants and predicates letters . We apply (D2) to wffs of this fragment of L'+ on the model of our application of (D2) to wffs of the propositional language L, making suitable adjustments to the notion of interpretations, partial interpretation, and α-relevant model ofα.[6]

We are now in a position to offer the following objectually based definition of content:

(D2.3) ß $\models_C \alpha$ $=_{df}$ α and ß are contingent, ß $\models \alpha$, and for any model M of ß and α, ß $\vdash_C \alpha$ relative to M.

I conjecture that (D2.3), or something very close to it, applied to wffs of L' is demonstrably equivalent to (D2) applied to wffs of L'.

5. A Model-Theoretic Definition of Content for L' with Identity

Gemes (1994) made fairly heavy weather of the question of how to define content for wffs of languages with an identity operator. However since we are

[6] Most importantly, full interpretations for this propositional fragment of L' are taken to be assignments of n-tuples of entities from the relevant domain D to each n-placed predicate of L' and assignments of individuals from D to each individual constant of L'. Interpretations here are strictly objectual rather than substitutional.

making use here of the resources of model theory rather than the purely syntactical resources of Gemes (1994) we have the means for a more smooth approach.

Let L'= be L' supplemented with the identity predicate '='. Then, given a little fiddling, (D2) can operate as a definition of content for wff of L'=. The notions of a full interpretation, extension, and an α-relevant model of α (Cf. (D2.1) above) can be defined as for L'. We need however to amend our definition of a partial interpretation and what it is for arbitrary atomic wff ß to be relevant to arbitrary wff α.

A partial interpretation of L'= is a valuation function which assigns to at least one atomic wff of L'= one of the two truth values true (T) and false (F) and assigns to no atomic wff of L'= both T and F and fulfills the following conditions regarding identity statements

(i) No identity statement $\ulcorner\alpha=\alpha\urcorner$ is assigned the value F.
(ii) For any identity statement $\ulcorner\alpha=ß\urcorner$, if $\ulcorner\alpha=ß\urcorner$ is assigned a value then $\ulcorner ß=\alpha\urcorner$ is assigned the same value.
(iii) For any pair of identity statements $\ulcorner\alpha=ß\urcorner$ and $\ulcorner ß=ø\urcorner$ if $\ulcorner\alpha=ß\urcorner$ and $\ulcorner ß=ø\urcorner$ are assigned T then so is $\ulcorner\alpha=ø\urcorner$.
(iv) If an atomic wff α is assigned a truth value V and ß is a constant that occurs in α, then for any individual constant ø, if $\ulcorner ß=ø\urcorner$ is assigned the value T, then the value V is assigned to any atomic wff α' which differs from α only in that one or more occurrence of the individual constant ß is replaced by ø.

An atomic wff ß is relevant to wff α iff for some partial interpretation P, every full interpretation P' that is an extension of P is a model of α and each such extension of P assigns the same value to ß, and there is no proper sub-interpretation P'' of P, such that for every full interpretation of P''' that is an extension of P'', P''' is a model of α.

Given these preliminaries,

(D2) ß $\models_c \alpha =_{df}$ α and ß are contingent, ß $\models \alpha$, and every α-relevant model of α has an extension which is a ß-relevant model of ß,

is applicable to wffs of L'=.

Some examples of content for wffs of L'=:

$$(x)Fx \models_c Fa_1 \qquad (x)Fx \not\models_c (x)(Fx \vee Gx)$$
$$Fa_1 \& (a_1=a_2) \models_c Fa_2 \qquad a_1=a_2 \not\models_c (Fa_1 \vee {\sim}Fa_2)$$
$$Fa_1 \& {\sim}Fa_2 \models_c a_1 \neq a_2 \qquad Fa_1 \not\models_c a_1=a_2 \rightarrow Fa_2$$

Note, 'Fa_1 is relevant to '$a_1=a_2 \rightarrow Fa_2$'. To see this consider that partial interpretation I that assigns T to the atomic wff 'Fa_1' and makes no other assignments. Every full interpretation that is an extension of I is a model for '$a_1=a_2 \rightarrow Fa_2$' and assigns the same value, namely T, to 'Fa_1, and I has in fact no proper sub-interpretations. Also, '$a_1=a_2$' is relevant to 'Fa_1&~Fa_2'. To see this consider that partial interpretation, call it I', that assigns the vale T to 'Fa_1' and the value F to 'Fa_2'. Now there is no proper sub-interpretation of I' such that every full extension of that sub-interpretation is a model of 'Fa_1&~Fa_2', and, further, every full extension of I assigns the same value, namely F, to '$a_1=a_2$'.

6. Alternative Notions I: Stelzner Consequences

The notion of content explored here and in Gemes (1994) is specifically designed to be employed in developing certain projects in the philosophy of science. Thus Gemes (1993) uses the notion of content to explicate the notion of hypothetico-deductive confirmation and the notion of natural axiomatizations; Gemes (1994a) uses it to clarify Friedman's notion of unification as reduction. Other non-standard notions of consequence have of course been introduced for other purposes.

In Gemes (1994) the content notion of consequence was briefly compared with Parry's notion of analytical entailment, Relevance logic entailment and Grimes's notion of narrow consequence. Relevance logics generally countenance 'pvq' as a consequence of 'p'. While analytical entailment does not hold between 'p' and 'pvq', it does hold between p's classical equivalent 'p&(pvq)' and '(pvq). Furthermore, 'p&q' analytically entails 'pv~q'. 'pv~q' also counts as a narrow consequence of 'p&q'. These and other results render the notions of analytical entailment, relevant entailment, and narrow consequence as less useful than the notion of content in serving those ends for which the notion of content was developed - for more on this see Gemes (1994).

We shall now consider a notion of consequence that is nearer to our above defined notion of content part.

As mentioned earlier, Gemes (1994) took definitions such as (D1) to give accounts of basic content rather than full content because they did not allow that disjunctions of content parts always count as content parts. For instance, while both the L wffs 'p' and 'q' count as content parts, by (D1) and (D2), of the L wff 'p&q', 'pvq' does not, according to (D1) and (D2), count as a content part of 'p&q'. Now in fact it is easy to construct an account which takes (non-tautologous) conjunctions of content parts as content parts. We need merely refine our notion of disjunctive normal form into the notion of a minimal disjunctive normal form. Thus where the dnf of '(r&(pvq)' is '(p&q&r)v(p&~q&r)v(~p&q&r)' its minimal disjunctive form, mdnf, is '(p&r)v(q&r)'.

The disjuncts of arbitrary (consistent) wff α's mdnf are each and every (consistent) conjunction ß of atomic wffs and negated atomic wffs, such that ß entails α and no proper sub-conjunction of ß entails α.[7] We might see an arbitrary disjunct ß of arbitrary α's dnf as specifying a world where α is true and where that world is specified only in terms of propositional variables relevant α. On the other hand we might see an arbitrary disjunct ß of arbitrary α's mdnf as specifying a minimal world for the truth of α.

Should we desire a definition of content that counts disjunctions of content parts as content parts we could simply replace the terms 'α_{dnf}' and '$ß_{dnf}$' in (D1) with the terms 'α_{mdnf}' and '$ß_{mdnf}$' where the later designate, respectively, the minimal disjunctive normal form of α and the minimal disjunctive form of ß. Thus we have the following definition for wffs of L,

(D3) ß $\vdash_S \alpha =_{df}$ (i) α and ß are contingent, (ii) ß $\vdash \alpha$, and (iii) each disjunct of α_{mdnf} is a sub-conjunction of some disjunct of $ß_{mdnf}$.

Something like this definition has been proposed independently in Stelzner (1992), Section 3.1, hence we use '$\vdash_S$' to designate the (D3) relation of being a Stelzner consequence.

While Stelzner (1992) provides a syntactical definition for a propositional language similar to L he does not propose any definition for quantificational languages, nor does he consider any model-theoretic analogs of his definition. However, we may apply (D3) to the quantificational language L' provided we again make recourse to the resources of propositional infinitary logics and form α_{mdnf} for arbitrary wff α of L' by putting Dev(α), as defined in Section 3. above, into minimal disjunctive normal form. This yields the result, for instance, that '(∃x)(FxvGx)' is a Stelzner consequence of '(x)(Fx&Gx)' since '(∃x)(FxvGx)'$_{mdnf}$ is '(Fa_1 v Ga_1 v Fa_2 v Ga_2 v .....Fa_n v Ga_n v .....)' and '(x)(Fx&Gx)'$_{mdnf}$ is '(Fa_1 & Ga_1 & Fa_2 & Ga_2 & .....Fa_n & Ga_n & .....)' and every disjunct of the former is a sub-conjunction of the one single disjunct of the later.

We may define a model theoretic analog of (D3) for both the propositional language L and the quantificational language L' by first defining the notion of a minimal model of arbitrary wff α as follows

(D4.1) P is an minimal model of wff $\alpha =_{df}$ P is a partial interpretation such that for any full interpretation P' that is an extension of P, P' is a

[7] Ideally, logically equivalent disjuncts would be eliminated in favor of some canonical equivalent. Thus mdnf of '(p&q)vr' would not be '(p&q)v(q&p)vr' but '(p&q)vr'. But nothing substantive hangs on this refinement.

model of α, and there is no proper-sub-interpretation P'' of P such that for every full interpretation P''' that is an extension of P'', P''' is a model α.

Note, where (D4.1) is applied to wffs of L, interpretations and partial interpretations are defined as in Section 2. above, that is, as assignments to the propositional variables of L. Where (D4.1) is applied to wffs of L', interpretations and partial interpretations are defined as in Section 4. above, that is, as assignments to (closed) atomic wffs of L.

Thus we have the model theoretic definition of Stelzner consequence:

(D4) ß $\models_S \alpha =_{df}$ α and ß are contingent, ß $\models \alpha$, and every minimal model of α has an extension which is a minimal model of ß.

We have a correctness and completeness proof for (D3) and (D4) which applies for both wffs of L and L'.

Theorem 9: For any wffs α and ß, ß $\models_S \alpha$ iff ß $\models_S \alpha$.

Proof: Cf. definitions (D3) and (D4) and note that (i) each minimal model of α uniquely characterizes and is uniquely characterized by a unique disjunct of α_{mdnf}, (ii) each minimal model of ß uniquely characterizes and is uniquely characterized by a unique disjunct of $ß_{mdnf}$ and (iii) a minimal model m of α has an extension which is a minimal model m' of ß iff the disjunct of α_{mdnf} which characterizes m is a sub-conjunction of the disjunct of $ß_{mdnf}$ that characterizes m'.

Proofs of theorems 2, 3, and 4 could be suitable amended to demonstrate that (D4) applied to both the propositional language L and the quantificational language L' is reflexive over contingent wffs of L and L', transitive for both L and L', and allows for substitution of logical equivalents.

(D4), and hence (D3), has the consequence that (non-tautologous) disjunctions of Stelzner consequences count as Stelzner consequences. Thus we have the theorem:

Theorem 10: If ß $\models_S \alpha$ and ß $\models_S \sigma$ and $\ulcorner(\alpha v \sigma)\urcorner$ is not a tautology then ß $\models_S (\alpha v ß)$.

Proof: Suppose ß $\models_S \alpha$ and ß $\models_S \sigma$ and $\ulcorner(\alpha v \sigma)\urcorner$ is not a tautology. Then α, σ and ß are contingent and ß $\models \alpha$ and ß $\models \sigma$. So ß $\models (\alpha v \sigma)$ and hence, since ß is not a contradiction, $\ulcorner(\alpha v \sigma)\urcorner$ is contingent. Now any minimal model for $\ulcorner(\alpha v \sigma)\urcorner$ is either an minimal model for α or an minimal model for σ, so given ß $\models_S \alpha$ and ß $\models_S \sigma$, any minimal model for $\ulcorner(\alpha v \sigma)\urcorner$ has an extension which is a minimal model for ß, so ß $\models_S(\alpha v\sigma)$.

The proof of Theorem 10 holds equally for (D4) applied to wffs of L and L'.

Every content-consequence of a given wff is a Stelzner consequence of that wff. To prove this for propositional language L we let ⌜At(α)⌝ designate the set of atomic wffs occurring in α and ⌜RAt(α)⌝ designate the set of atomic wffs that are relevant to α and make use of the following lemma derivable from Theorem 13 in the appendix below and Theorem 1 of Gemes (1994);

Lemma 1: For any L wffs α and ß, ß $\vdash_C\alpha$ by (D1) iff α and ß are contingent and ß $\vdash\alpha$ and there is some σ such that σ⊣ ⊢α and there is no ø such that ß ⊢ø ⊢σ, σ ⊬ø and At(ø) ⊆At(α).

Theorem 11: For any L wffs α and ß, if ß $\vdash_C\alpha$ then ß $\vdash_S\alpha$

Proof: We prove Theorem 11 by reductio. Suppose ß $\vdash_C\alpha$ and ß $\nvdash_S\alpha$ Then ß ⊢α. Let $ß_j$ be any disjunct of $ß_{mdnf}$. Then $ß_j$ ⊢ß, so $ß_j$ ⊢α. Since $ß_j$ is a disjunct of $ß_{dnf}$, $ß_j$ is a conjunction of atomic wffs and negated atomic wffs, and hence, since $ß_j$ ⊢α, there must be at least one minimal sub-conjunction B_k of $ß_j$, such that $ß_k$ ⊢α. So $ß_k$ is a disjunct of α_{mdnf}. So each disjunct of $ß_{mdnf}$ entails some disjunct of α_{mdnf}. Let α* be the conjunction of all the disjuncts α_k of α_{mdnf} such that for some disjunct $ß_j$ of $ß_{mdnf}$, $ß_j \vdash_k$. So $ß_{mdnf}$ ⊢α*. Now since α* is just a sub-disjunction of $_{mdnf}$, α ⊢α* $\vdash\alpha_{mdnf}$. Since, ß $\nvdash_S\alpha$ there is some disjunct α_S of $\alpha\alpha_{mdnf}$ such that for every disjunct $ß_j$ of $ß_{mdnf}$, $ß_j \nvdash\alpha_S$. So α* does not contain α_S as a disjunct. Now suppose for some disjunct α_X of α*, $\alpha_S \vdash\alpha_X$. Then since α_S and α_X are both disjuncts of α_{mdnf} $\alpha_X \vdash\alpha_S$, otherwise α_X would be a proper sub-disjunction of α_S, in which case, since $\alpha_X \vdash\alpha$, α_S could not be a disjunct of α_{mdnf}. So for every disjunct α_X of α*, $\alpha_S \nvdash\alpha_X$. So $\alpha_S \nvdash\alpha$*. Now since α_S is a disjunct of α_{mdnf}, $\alpha_S \vdash\alpha_{mdnf}$. So $\alpha_{mdnf} \nvdash\alpha$*, so α ⊬α*. Since $ß_{mdnf}$ ⊢α*, ß ⊢α*. So ß ⊢α* ⊢α and α ⊬α*. Now consider any σ such that σ⊣ ⊢α. Since α* is a sub-disjunction of α_{mdnf}, and At(α_{mdnf}) ⊆ At(α_{dnf}) and At(α_{dnf}) = RAt(α), At(α*)⊆ RAt(α). Now since σ⊣ ⊢α, RAt(α) = RAt(σ). So At(α*)⊆ RAt(σ). And since RAt(σ) ⊆ At(σ), At(α*)⊆ At(σ). So for any σ⊣ ⊢α, ß ⊢α* ⊢σ, σ ⊬α* and At(α*)⊆At(σ). So, by Lemma 1, ß $\nvdash_C\alpha$.[8]

The converse of Theorem 11 does not hold. The following are cases of Stelzner consequences that do not count as content consequences,

p&q $\vdash_S$pvq p&q $\nvdash_C$pvq

[8] This proof would also suffice to show that for wffs of L', if ß $\vdash_C\alpha$ then ß $\vdash_S\alpha$ provided that a analog of Lemma 1 could be proven for wffs of L' and '⊢' is given a "substitutional reading".

$$(x)Fx \vdash_S (\exists x)Fx \qquad (x)Fx \not\vdash_C (\exists x)Fx$$
$$p \equiv q \vdash_S p \rightarrow q \qquad p \equiv q \not\vdash_C p \rightarrow q$$

For certain purposes the content relationship seems more useful then the Stelzner consequence relationship. Forexample, suppose we are trying to define the notion of partial truth.[9] If we say the 'p≡q' is partially true where it has a true Stelzner consequence then we get the result that where 'p' is false and 'q' is true then 'p≡q' is partially true, since under these circumstances 'p≡q's Stelzner consequence 'p→q' is true. Indeed, where partial truth is defined as having a true Stelzner consequence, 'p≡q' is always partially true since it is always the case that one of its Stelzner consequences 'p→q' or 'q→p' is true.

Stelzner himself is particularly concerned with applications to Deontic logic. For instance, where '*O*' is an obligation operator he wishes to avoid the inference from '*O*(p)' to '*O*(pvq)'. Part of the reason for avoiding this inference is that it leads to problems with the notion of derived obligations.

However it seems plausible to say that one might equally reject the inference from '*O*(p≡q)' to '*O*(˜pvq)'. For instance, one might question the move from 'You ought to give your niece a Christmas present if and only if you give your nephew a Christmas present' to 'You ought to not give your niece a Christmas present or give your Nephew a Christmas present.

Indeed, Stelzner develops his notion of consequence in order to preserve the condition (Stelzner 1992, p.196),

(PRC) Positive Relevance Condition: Minimally (at least partially) fulfilling a derived norm implies minimally (at least partially) fulfilling a basic norm.

Yet clearly if I give my nephew a Christmas present (NE) and do not give my niece one (~NI), thus fulfilling the norm O(~NIvNE) derived from the basic norm O(NI≡NE), then I have not partially fulfilled that basic norm. Furthermore, under the Stelzner notion of consequence, no mater what I do I will be at least partially fulfilling the basic norm O(NI≡NE) since whatever I do will fulfill at least one of the two derived norms O(˜NIvNE) and O(Niv˜NE).

Despite these differences, by and large, the content notion of consequence and the Stelzner notion of consequence are so close together that for many purposes they serve equally well.

7. Alternative Notions 2: Relevant Consequences

Paul Weingartner and Gerhard Schurz, individually and often together, have

[9] This is not a purpose suggested in Stelzner (1991).

written a series of illuminating articles, for example, Weingartner and Schurz (1986), (1987), Schurz (1991), (1991a), Weingartner (1988), exploring a family of consequence relations that share the content relationships strictures about the transition from wff α to the disjunction $\ulcorner(\alpha \vee ß)\urcorner$. While Schurz and Weingartner have proposed various non-equivalent consequence relations, for our purposes it will suffice to concentrate on what Schurz calls the relation of being a relevant consequence, which we symbolize as '$ß \vdash_r \alpha$' and define as follows,

(D5) $ß \vdash_r \alpha =_{df} ß \vdash \alpha$ and there is no n-ary predicate in α that is replaceable in some of its occurrences in α by a new predicate of the same arity (not occurring in α or ß) salva validitate of $\alpha \vdash ß$.

For the purposes of applying (D5) to wffs of propositional languages we can take propositional constants such as 'p' to be 0-ary predicates.

The relation of being a relevant consequence is not reflexive, for instance, 'pvp' is not a relevant consequence of itself. Nor is it transitive, for instance, '(pvq)&(pvr)' is a relevant consequence of 'pv(q&r)' which itself is a relevant consequence of '(pvq)&r', but '(pvq)&(pvr)' is not a relevant consequence of '(pvq)&r'. Nor is it closed under substitution of classical logical equivalence, for instance, 'p' is a relevant consequence of 'p' but 'p's classical logical equivalent 'p&(pvq)' is not a relevant consequence of 'p'. The relation of being a relevant consequence is not conservative under the addition of logical operators to a language. For instance, while in $L_{\sim\vee\&\rightarrow}$, '(p→q)&(q→p)' is not a relevant consequence of 'p&q', in $L_{\sim\vee\&\rightarrow\equiv}$, where $\ulcorner\alpha\equiv ß\urcorner$ is defined as abbreviating $\ulcorner(\alpha\rightarrow ß)\&(ß\rightarrow\alpha)\urcorner$, '(p≡q)' is, according to (D5), a relevant consequence of 'p&q'.[10] For this later reason Schurz and Weingartner generally work with languages not including a material equivalence operator.[11] The relevant consequence relation does not lend itself to any transparent model-theoretic analysis in the manner, for example, of our (D2) definition of the content relationship.[12]

Many content consequences count as relevant consequences, for instance, for each of the following pairs of wffs the second counts as both a relevant consequence and a content consequence of the first:

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10 As noted out by a referee from this journal, Schurz (1991), p.415, points out that this true only for the addition of logical equivalence and exclusive disjunction and that the concept of relevant consequence is invariant with respect to any subset of the primitive operators $\{\sim, \vee, \&, \rightarrow\}$.

11 Such an operator could of course be countenanced provided before determining relevant consequences of arbitrary wff α via (D5) all occurrences of '≡' were eliminated in favor of constructions in terms of, say, '→'.

12 Schurz (1991a, p.77) does in fact offer a semantics for relevant implication after observing that "[t]his semantics cannot be the usual one. This follows from the fact that R1-relevant implication is not closed under substitution."

<(p&q),p>, <(p&(rv~r),p>, <((p&q)v(r&s)),(pvr)>, <(~p&(pvq)),q>, <(x)Fx,Fa>, <(x)(Fx&Gx),(x)Fx>, <(Fa&(a=b)),Fb>, <(Fa&~Fb),(a≠b)>.

For many pairs of wffs while the second is a classical consequence of the first it is neither a content consequence or a relevant consequence of the first, for instance:

<p,(pvq)>, <(p&(pvq)),(pvq)>, <(p&q),(pvq)>, <q,(p→q)>, <~p, (p→q)>, <(x)Fx,(x)(FxvGx)>.

There are cases of content consequences that do not count as relevant consequences. These invariably involve wffs with redundant sub-formulas or redundant occurrences of a given vocabulary. Thus consider the following pairs where the second counts as content consequence but not a relevant consequence of the first:

<p,(p&(pvq)>, <p,(pvp)>

Where a content consequence is free of all such redundancies it is a relevant consequence. Now the dnf of any wff is in fact free of all such redundancies. Thus we have the theorem,

Theorem 12: For any wffs α and ß of L, if ß $\vdash_c \alpha$ then for some σ, $\sigma \dashv\vdash \alpha$ and ß $\vdash_r \sigma$.

Proof: Suppose ß $\vdash_c \alpha$. Then ß $\vdash \alpha_{dnf}$. Now we show thαat if one or more occurrence of a propositional variable n α_{dnf} is replaced with a propositional variable not occurring in either α_{dnf} or ß then ß does not entail the resultant formula. Thisby (D5) suffices to show that ß $\vdash_r \alpha_{dnf}$. Let ø be any propositional variable in α_{dnf}. Let σ be any propositional variable that occurs in neither α_{dnf} or ß. Let α_i be any disjunct of α_{dnf}. Let α_{dnf}/α_i/ø:σ be the wff that results from replacing the occurrence of ø (or ~ø) in α_i in α_{dnf} by σ and let α_i/ø:σ be the wff that results from replacing the occurrence of ø (or ~ø) in α_i by σ. Now let $ß_j$ be one of the disjuncts of ß such that $ß_j \vdash \alpha_i$. Now $\alpha_i \vdash$~$(\alpha_{dnf} - \alpha_i)$, where $(\alpha_{dnf} - \alpha_i)$ is α_{dnf} without the disjunct α_i. So $ß_j \vdash$~$(\alpha_{dnf} - \alpha_i)$. Now since ~σ $\vdash$~(α_i/ø:σ), ß&~σ $\vdash$~$(\alpha_{dnf} - \alpha_i)$&~($\alpha_i$/ø:σ). But ~$(\alpha_{dnf} - \alpha_i)$&~($\alpha_i$/ø:σ) is just ~$\alpha_{dnf}/\alpha_i$/ø:σ. So ß $\vdash$~α_{dnf}/α_i/ø:σ. So ß $\vdash\_/\alpha_{dnf}/\alpha_i$/ø:σ.

Theorem 12 holds also for wffs of L'. To prove this we would need to again avail ourselves of the resources of the propositional fragment of infinitary logics and construe '$\vdash$' substitutionally and replace reference to propositional variables in the proof with reference to (closed) atomic wffs

What theorem 12 tells us is that where α is a content part of ß there is some σ logical equivalent to α such that σ is a relevant consequence of ß. There are cases of relevant consequence that do not count as content consequences. The following are some noteworthy cases:

Case 1. $(p\&q)v(\sim p\&\sim q) \vdash_r (pv\sim q)$
Case 2. $(pvq)\&r \vdash_r pv(q\&r)$
Case 3. $(p\rightarrow q)\&(q\rightarrow p)\&(s\rightarrow p)\&(r\rightarrow q) \vdash_r (s\rightarrow q)\&(r\rightarrow p)$
Case 4. $Fa \vdash_r (\exists x)Fx$
Case 5. $(x)Fx \vdash_r (\exists x)Fx$
Case 6. $Fa \vdash_r a=b\rightarrow Fb$.[13]

Note, '$(p\&q)v(\sim p\&\sim q)$' in Case 1 and '$(p\rightarrow q)\&(q\rightarrow p)$' in Case 3 can both be replaced by '$p\equiv q$' in languages with the material equivalence operator '$\equiv$'. Also, each of cases 1-3 have various quantificational analogs. For instance, as a quantificational analog of Case 3 we have,

Case 3a. $(x)(Rx\equiv Sx)\&(x)(Px\rightarrow Rx)\&(x)(Hx\rightarrow Sx) \vdash_r (x)(Hx\rightarrow Rx)\&(x)(Px\rightarrow Sx)$.

These type of cases provide divergent results in applications to various problems.

Verisimilitude: Schurz and Weingartner have employed the notion of relevant consequence in order to rehabilitate Popper's definition of verisimilitude. Gemes (1991) employs the notion of content parts for similar ends.

Let $\ulcorner X_T \urcorner$ name what Poppers calls the truth content of T, that is, the set of all true (classical) consequences of theory X, and $\ulcorner X_F \urcorner$ name what Popper calls the falsity content of T, that is, the set of all false (classical) consequences of theory X. Then the following is Popper's definition of verisimilitude:

> **(D5) Assuming that truth-content and falsity-content of two theories T1 and T2 are comparable, T2 has more verisimilitude than T1 iff**
>
> **(a)** $\mathbf{T1_T \subseteq T2_T \;\&\; T2_F \subset T1_F}$
>
> **or**
>
> **(b)** $\mathbf{T1_T \subset T2_T \;\&\; T2_F \subseteq T1_F}$

Pavel Tichy (1974), John Harris (1974), and David Miller (1974), have shown that (D5) has the consequence that if T2 is false than for any T1, T2 is not closer to the truth than T1. As if this result were not bad enough, we should also note that

[13] An unnamed referee from this journal informed me that in various manuscripts Schurz eliminates '=' in favor of certain congruence relations, in which case 'a=b□Fb' does not counts as a relevant consequence of 'Fa'. Schurz himself reports that this is true of his Habilitation, Schurz (1989), and in a book manuscript currently under review.

(D5) has the horrendous consequence that for any (finitely axiomatizable) T1, true or false, if T1 does not entail every truth then T1 does not have more verisimilitude than its negation - which, for convenience, we symbolize as '~T1'.[14]

Proof: Let T1 be any arbitrary (finitely axiomatizable) theory such that there is some truth t such that $T1 \nvdash t$. Now consider $\ulcorner(t \vee \sim T1)\urcorner$. Since t is true $\ulcorner(t \vee \sim T1)\urcorner$ is true. So clearly $(t \vee \sim T1) \in \sim T1_T$. Yet $(t \vee \sim T1) \notin T1_T$, otherwise $T1 \vdash (t \vee \sim T1)$, and hence $T1 \vdash t$, contra our choice of t. Therefore $\sim T1_T \subseteq T1_T$. Therefore T1 does not have more verisimilitude than ~T1.

Both the Tichy-Harris-Miller result and the result demonstrated above can be avoided if the definition of the truth (falsity) content of a theory is framed in terms of true (false) relevant consequences or true (false) content consequences. For instance, if one does not allow true but non-relevant consequences into the truth content of a theory the above proof is blocked since $\ulcorner(t \vee \sim T1)\urcorner$ would not then be part of the truth content of ~T1. Similarly, the restriction of the truth content to true content consequences also eliminates $\ulcorner(t \vee \sim T1)\urcorner$ as part of the truth content of ~T1.

However, where one uses the notion of relevant consequences to rehabilitate the definition of verisimilitude one gets various undesirable results.[15]

For instance, one gets the result that any basic wff of the form $\ulcorner\sigma\phi\urcorner$ or $\ulcorner\sim\sigma\phi\urcorner$, where σ is a predicate and ø is a individual constant of the relevant language, does not have any more verisimilitude than its negation, provided at least one thing lacks ø and one thing has ø. For example, 'Boston is in America', symbolically 'Ab', has no more verisimilitude than its negation, '~Ab', since the later but not the former has as part of its truth content 'Something is not in America', '(∃x)~Ax)'.

For instance, one gets the result that, for any atomic wffs 'p' and 'q', where 'p' is true and 'q' is false, the true claim 'p≡~q' does not have more verisimilitude than the false claim 'p≡q'. This is so because were truth content is defined in terms of true relevant consequences and 'p' is true 'pv~q' counts as part of the truth content of 'p≡q' but not part of the truth content of 'p≡~q'.Thus one gets the result that the claim 'Carter was President of America if and only if it is not the case that Gorbachev was Prime Minister of England' has no more verisimilitude than the claim 'Carter was President of America if and only if Gorbachev was

[14] Where T1 is a finitely axiomatizable theory we can understand '~T1' as the negation of the conjunction of all the axioms of T1' where T1' is a finite axiomatization of T1.

[15] In their various reformulations of the definition of verisimilitude, for instance, in Schurz (1991) and Weingartner and Schurz (1987), Weingartner and Schurz make recourse to the notion of relevant consequent elements which in turn is defined in terms of relevant consequences. However this refinement does not affect any of the counter-examples dealt with here.

Prime Minister of England'.[16]

For instance, one gets the result that where atomic wff 'Fa' is true, 'a=b' is false and no individual of the relevant domain lacks the property F, 'Fa' does not have any more verisimilitude than `˜Fa' since '˜Fa' has as part of its truth content 'a=b →˜Fb' which is not part of the truth content of 'Fa'.Thus one gets the result that where the domain is the set of all physical objects the claim, 'Carter has mass' has no more verisimilitude than 'It is not the case that Carter has mass'.[17]

These unfortunate results are avoided if truth content is defined in terms of content consequences rather than relevant consequences.

Hypothetico-Deductive Confirmation

Schurz (1991), Schurz and Weingartner (1986), Gemes (1990), Grimes (1990), and others have noted that classical expressions of hypothetico-deductivism, hereafter H-D, such as

(H-D1) e (directly) confirms T iff T entails e,

yield the result that arbitrary disjunctions may be used to confirm theories.[18] For instance, suppose one has just observed that Sydney has a harbor bridge. From this one deduces that 'Sydney has a harbor bridge or all the planets travel in elliptical orbits' is true which, according to (H-D1), confirms the claim 'All the planets travel in elliptical orbits''. Where H-D is framed in terms of relevant or content consequences this type of result can be avoided.

However, framed in terms of relevant consequences H-D produces various undesirable results.[19]

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[16] One might claim that where 'p' is true and q is false 'p→~q' does not have any more verisimilitude than 'p≡q' since it would be theory progress if 'p≡q's true relevant consequence 'q→p' was added to the true 'p≡~q' thus yielding the new truth 'p'. But then, by the same token, for any true 'p' and 'q', one might claim 'p' has no more verisimilitude than '~p' since adding '~p's true consequence '~pvq' to the true 'p' yields the new truth 'q'.

[17] Of course where the language in question eschews '=' in favor of a congruence relationship this criticism does not apply (Cf. note 14 above).

18 Schurz (1991) and Weingartner and Schurz (1986) claim that the variant of (H-D) they consider has the result that any true contingent e confirms any contingent h. To get this result they appeal to the "condition of strengthening the confirmans": if A confirms T and A' is consistent, true (accepted) and logically implies A, then also A' confirms T. However this principle is highly suspect, yielding the result, for instance, that given that 'Die A came up 5' confirms 'Neither Die A nor Die B came up 6', 'Die A came up came up 5 and die B came up 5 or 6' confirms 'Both Die A and Die B came up 1,2,3,4, or 5'.

[19] Schurz (1991) and Weingartner and Schurz (1986) actually suggest formulating H-D in

For instance, H-D reformulated in terms of relevant consequences yields the result that observing something that is F confirms the claim 'Fa' since '(∃x)Fx' is a relevant consequence of 'Fa'. Thus, observing that something is President of Russia confirms the claim that Carter is president of Russia. Note, while H-D reformulated in terms of relevant consequences yields the result that '(∃x)Fx' confirms 'Fa' it does not yield the result that the stronger 'Fa v Fb' also confirms 'Fa'. This seems somewhat paradoxical and indeed violates a constraint on hypothetico deductive confirmation advocated by Schurz himself, namely, that if S hypothetico-deductively confirms T and S* logically implies S and is consistent with T then S* also confirms T.[20]

Another undesirable result is that one confirms that a lacks property F by observing that a is not identical to b or lacks F since 'a≠bvFb' is a relevant consequence of 'Fa'.[21] So observing that Major is not identical to Clinton or Major is not President of America allows one to confirm that Clinton is not President of America.

A further undesirable result is that '(x)(Hx→Rx)&(x)(Px→Sx)' confirms '(x)(Rx≡Sx)&(x)(Px→Rx)&(x)(Hx→Sx)' since the later is a relevant consequence of the former. So, the conjunction of the claims 'All humans are rational animals' and 'All planets have a roughly spherical shape' confirms the conjunction of 'Anything is a rational animal if and only if it is roughly spherical in shape' and 'All planets are rational' and 'All humans are roughly spherical in shape'. More generally, one has the result that one can confirm that there is a linkage between qualities A and C and a linkage between qualities B and D and a linkage between qualities C and B by observing a linkage between A and B and a linkage between C and D since '(x)(Ax≡Bx)&(x)(Cx≡Dx)' is a relevant consequence of '(x)(Ax≡Cx)&(x)(Bx≡Dx)&(x)(Cx≡Bx)'. So, for instance, finding that being human and being rational are (materially) equivalent and being water and being H_2O are (materially) equivalent confirms that being human and being H_2O are (materially) equivalent and being rational and being water are (materially) equivalent and being H_2O and being rational are (materially) equivalent.

Where H-D is framed in terms of content consequences none of these results is obtainable.

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terms of inferences with relevant premises as well are relevant conclusions. Furthermore, Schurz (1994) in response to Gemes (1994), makes recourse to the notion of relevant consequence elements. However none of these details affect the problems discussed below.

[20] For more on the pros and cons of this constraint see Schurz (1994) and Gemes (1994b). For more on the problems of combining this constraint with H-D see Gemes (1996).

[21] Of course where the language in question eschews '=' in favor of a congruence relationship this criticisms do not apply (Cf. note 13 above).

Gricean Conversational Maxims: There are applications for which the notion of relevant consequence seems prima facie better suited than the notion of content consequence. For instance, in trying to capture the Gricean maxim of speaking informatively it seems more appropriate to enjoin

(I) Utter only relevant consequences of your beliefs

than

(II) Utter only content consequences of your beliefs.

Both (I) and (II) militate against my uttering the claim 'My brother is in Sydney or New York' when in fact I believe him to be in New York. However where I believe both that my brother is in Sydney and that he as two children only (I), but not (II), militates against my uttering the claim 'My brother is in Sydney AND he has two children or he does not have two children'.

In this case it is the content consequences property of allowing for substitution of classical logical equivalents that causes problems. While in the case just described the uttering of 'My brother is in Sydney' is Grice-admissible, the uttering of its classical logical equivalent 'My brother is in Sydney AND he has two children or he does not have two children' is presumably not Grice-admissible. Earlier we noted that the notion of relevant consequence, unlike the notion of content consequence, does not allow for any redundancies. It is this feature that makes the notion of relevant consequence prima facie more suitable than that of content consequence for capturing Gricean maxims.[22]

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[22] However, as noted in footnote 4. above, we may construct a notion of content that does not treat logical equivalents equivalently. Such a notion of content would be a serious competitor with the notion of relevant consequence for application to the problem of formalizing Gricean conversational maxims. Thus suppose, for instance, that α_{dnf} was defined in terms of the full vocabulary of α rather than merely the essential vocabulary of α. Then the dnf form of '(p&(qv~q)' would be '(p&q)v(p&~q)' and the dnf of '(p&q)' would be '(p&q)'. So, applying this definition of dnf to (D1) we get the result that while 'p' is a content part of '(p&q)', its logical equivalent '(p&(qv~q)' is not a content part of '(p&q)'. Alternatively, we could simply try to capture the Gricean norm by specifying

(III) Utter only concise content consequences of your beliefs.

A content consequence α of a belief set is concise iff there is no wff $\int$, such that $\int$ is logically equivalent to α and $\int$ is shorter than α. Note, both (II) and (III), unlike (I) militate against uttering "Jones should give his daughter a Christmas present or not give his son one" where one believes "Jones should give his daughter a Christmas present if and only if he gives his son one". Similarly, both (II) and (III), but not (I), militate against uttering 'Someone owns a Ford' where one believes 'Jones owns a Ford', or uttering 'Someone should exercise their right to vote' where one believes 'Everyone

Both the content notion of consequence and the notion of relevant consequence represent restrictions of the classical notion of consequence; they are partitions of the classical consequence relation. Which is preferable is best answered with respect to the particular applications one has in mind.[23]

should exercise their right to vote'.

[23] This paper has been greatly improved through the comments and criticisms provided by an unnamed referee from this journal.

Appendix

We now show that the following definition of content for wffs of L:

(D1*) $ß \vdash_c \alpha =_{df}$ (i) α and ß are contingent, (ii) $ß \vdash \alpha$, and (iii) there is no μ such that μ is a proper sub-disjunction of α_{dnf} and $ß \vdash \mu$,

given as BCPL1 in Gemes (1994), is equivalent to our definition for wffs of L

(D1) $ß \vdash_c \alpha =_{df}$ (i) α and ß are contingent, (ii) $ß \vdash \alpha$, and (iii) each disjunct of α_{dnf} is a sub-conjunction of some disjunct of $ß_{dnf}$

given above.

Before proving that (D1*) and (D1) are equivalent we need to consider some lemmas. In the following lemmas we use ⌜At(σ)⌝ to signify the set of atomic wffs occurring in wff σ

Lemma 13.1 $ß \vdash_c \alpha$ by (D1*), $At(\alpha_{dnf}) \subseteq At(ß_{dnf})$.

Proof: Cf.Lemma 2.1 of Gemes (1994).

Lemma 13.2 For any (contingent) wff α, and any disjunct α_j of α_{dnf}, $At(\alpha_j) = At(\alpha_{dnf})$.

Proof: Cf. the construction of α_{dnf}.

Lemma 13.3 Where $ß \vdash_c \alpha$ by (D1*), for any disjunct α_k of α_{dnf} and any disjunct $ß_j$ of $ß_{dnf}$, $At(\alpha_k) \subseteq At(ß_j)$.

Proof: This follows from lemmas 13.1 and 13.2.

Lemma 13.4 For any wffs σ and μ, where σ and μ are conjunctions of atomic wffs and negated atomic wffs if $At(\sigma) \subseteq At(\mu)$ and σ is not a sub-conjunction of μ then $\sigma \vdash {\sim}\mu$.

Proof: Assume σ and μ are conjunctions of atomic wffs and negated atomic wffs, $At(\sigma) \subseteq At(\mu)$, and σ is not a sub-conjunction of μ.

Since σ and μ are conjunctions and σ is not a sub-conjunction of μ, σ contains some conjunct α that is not a conjunct of μ. Now since σ is a conjunction of atomic and negated atomics α is an atomic wff or a negated atomic wff. Let ß be the atomic wff that occurs in α, in other words α is ß or α is ⌜~ß⌝. Now since $At(\sigma) \subseteq At(\mu)$, the atomic wff ß occurs in μ.

Now suppose α is ß. Then ß is not a conjunct of μ, since α is ex hypothesi not a conjunct of μ. Now since the atomic wff ß occurs in μ and μ is just a conjunction of atomic wffs and negated atomic wffs then where ß is not a conjunct of μ, ⌜~ß⌝ is a conjunct of μ. So where α is ß, σ $\vdash$ß and μ $\vdash$~ß, so σ $\vdash$~μ.

Now suppose α is ⌜~ß⌝. Then ⌜~ß⌝ is not a conjunct of μ, since α is ex hypothesi not a conjunct of μ. Now since the atomic wff ß occurs in μ and μ is just a conjunction of atomic wffs and negated atomic wffs then where ⌜~ß⌝ is not a conjunct of μ, ß is a conjunct of μ. So where α is ⌜~ß⌝, σ $\vdash$~ß and μ $\vdash$ß, so σ $\vdash$~μ.

Lemma 13.5 Where s is a proper sub-disjunction of α_{dnf} then where α_k is disjunct of α_{dnf} that is not a disjunct of s then $\alpha_k \vdash$~s.

Proof: Assume s is a proper sub-disjunction of α_{dnf} and α_k is disjunct of α_{dnf} that is not a disjunct of s. Now let α_j be any disjunct of α_{dnf} other than α_k. Then since each disjunct of α_{dnf} is a conjunction of atomic wffs and negated atomic wffs and each disjunct contains the same atomic wffs and each disjunct is unique, $\alpha_k \vdash$~α_j. So since s is just a disjunction of such α_j, $\alpha_k \vdash$~s.

We are now in a position to prove:

Theorem 13: ß $\vdash_c \alpha$ by (D1) iff ß $\vdash_c \alpha$ by (D1*).

Proof: We assume α and ß are contingent and ß $\vdash \alpha$ otherwise the case is trivial.

Now we show that if ß $\vdash_c \alpha$ by (D1*) then ß $\vdash_c \alpha$ by (D1) by reductio. Suppose ß $\nvdash_c \alpha$ by (D1) and ß $\vdash_c \alpha$ by (D1*). Let (α_1v...vα_m) be α_{dnf} and (ß$_1$v...vß$_m$) be ß$_{dnf}$. Then since ß $\nvdash_c \alpha$ by (D1) for some disjunct α_k of (α_1v...vα_m) there is no disjunct ß$_j$ of (ß$_1$v...vß$_m$) such that α_k is a sub-conjunction of ß$_j$. Now since ß $\vdash_c \alpha$ by (D1*), for each disjunct ß$_j$ of (ß$_1$v...vß$_m$), At(α_k)□At(ß$_j$) - cf. lemma 13.3. Now for any disjunct ß$_j$ of (ß$_1$v...vß$_m$), ß$_j$ is just a conjunction of atomics wffs and negated atomic wffs and α_k, being a disjunct of a α_{dnf}, is also just a conjunction of atomics wffs and negated atomic wffs, so since At(α_k)$\subseteq$ At(ß$_j$) and α_k is not a sub-disjunction of ß$_j$, $\alpha_k \vdash$~ß$_j$ - cf. lemma 13.4. So $\alpha_k \vdash$~(ß$_1$v...vß$_m$). So $\alpha_k \vdash$~ß. So ß $\vdash$~α_k. Now since ß $\vdash \alpha$, ß $\vdash$(α_1v...vα_m). Now since α_k is a conjunct of (α_1v...vα_m) and ß $\vdash$~α_k and ß $\vdash$(α_1v...vα_m), ß $\vdash$(α_1v..vα_{k-1}vα_{k+1}v...vα_m) - that is (α_1v...vα_m) less its disjunct α_k. Now since each disjunct of α_{dnf} is unique (α_1v..vα_{k-1}vα_{k+1}v...vα_m) is a proper sub-disjunction of (α_1v...vα_m). So ß $\nvdash_c \alpha$ by (D1*) contra our initial supposition.

Now we show that if ß $\vdash_c \alpha$ by (D1) then ß $\vdash_c \alpha$ by (D1*) by proving its contrapositive. Suppose ß $\nvdash_c \alpha$ by (D1*). Then for some proper sub-disjunction s of α_{dnf}, ß $\vdash$s. Let α_k be some disjunct of α_{dnf} that is not a disjunct of s. Then $\alpha_k \vdash$~s - cf lemma 13.5. So $\alpha_k \vdash$ ~ß and for each disjunct $ß_j$ of $ß_{dnf}$, $ß_j \vdash$ß, so $\alpha_k \vdash$ ~$ß_j$. Now since α_k is consistent and for each disjunct $ß_j$ of $ß_{dnf}$, $\alpha_k \vdash$~$ß_j$, α_k is not a sub-conjunction of any disjunct of $ß_{dnf}$. So ß $\not\vdash_c \alpha$ by (D1).

<u>References</u>

Gemes, K. (1990), "Horwich, Hempel, and Hypothetico-Deductivism", *Philosophy of Science*, **57**: 609-702.

_____. (1991), "Content Confirmation", Ph.D. Dissertation University of Pittsburgh.

_____. (1993), " Hypothetico-Deductivism, Content, and & The Natural Axiomatization of Theories", *Philosophy of Science*, **60**: 477-487.

_____. (1994), "A New Theory of Content", *Journal of Philosophical Logic*, **23**: 596-620.

_____. (1994a), "Explanation, Unification, and Content", *Nous*, **28**:225-240.

_____. (1994b), "Schurz On Hypothetic-Deductivism", *Erkenntnis*, **41**: 171-181.

_____. (1996), "Hypothetico-Deductivism: The Current State of Play; The Criteria of Empirical Significance: Endgame", forthcoming.

Grimes, T. (1990), "Truth, Content And the Hypothetico-deductive Method", *Philosophy of Science*, **57**: 514-522.

Harris, J. (1974), "Popper's Definitions of 'Verisimilitude'", *British Journal for the Philosophy of Science*, **25**:160-165.

Miller, D. W. (1974), "Popper's Qualitative Theory of Verisimilitude", *British Journal for the Philosophy of Science*, **25**:166-177.

Schurz, G. (1989), *Relevant Deduction in Science and Ethics. With a Case Study of the Is-Ought Problem*, Habilitationsschrift, Salzburg.

_____. (1991), "Relevant Deduction", in *Erkenntnis*, **35**:391-437.

_____. (1991b), "Relevant Deductive Inference", in *Advances in Scientific Philosophy*, (Eds.) G. Schurz and J. Dorn, Rodopi, Amsterdam.

_____. (1994), "Relevant Deduction and Hypothetico-Deductivism: A reply to Gemes", *Erkenntnis*, **41**: 183-188.

Stelzner, W. (1992), "Relevant Deontic Logic", *Journal of Philosophical Logic*, **21**: 193-216.

Tichy, P. (1974), "On Popper's Definitions of Verisimilitude", *British Journal for the Philosophy of Science*, **25**:155-160.

Weingartner, P. and Schurz, G. (1986), "Paradoxes Solved by Simple Relevance Criteria", *Logique et Analyse*, **29**: 3-40.

_____. (1987), "Verisimilitude Defined by Relevant Consequence Elements. A New Reconstruction Of Popper's Idea", in T. A. Kuipers (ed.) *What Is Closer-To-The-Truth?*, Poznan Studies in the Philosophy of Science and the Humanities, Vol. 10, Rodopi, Amsterdam.

Weingartner, P. (1988), "Remarks on the Consequence-Class of Theories", in E. Scheibe (ed.) *The Role of Experience in Science*, De Gruyter, Berlin/New York.

Endnotes