Coherence, Belief Expansion and Bayesian Networks

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Abstract

We construct a probabilistic coherence measure for information sets which determines a partial coherence ordering. This measure is applied in constructing a criterion for expanding our beliefs in the face of new information. A number of idealizations are being made which can be relaxed by an appeal to Bayesian Networks.

Introduction

Suppose that one receives information \( \{R_1, \ldots, R_n\} \) from \( n \) independent but less than fully reliable sources. Is it rational to believe this information? Following a tradition in epistemology that goes back to John Locke, we let belief correspond to a sufficiently high degree of confidence (Foley 1992, Hawthorne and Bovens 1999). There are three factors that determine this degree of confidence: (i) How surprising is the information? (ii) How reliable are the sources? (iii) How coherent is the information? First, suppose that the sources are halfway reliable and the information is halfway coherent. Then certainly the degree of confidence will be greater when the reported information is less rather than more surprising. Second, suppose that the information is halfway surprising and is halfway coherent. Let truth-tellers provide fully reliable information and let randomizers flip a coin for each proposition to determine whether they will affirm or deny it. Then certainly the degree of confidence will be greater when the sources are more like truth-tellers than when they are more like randomizers. Third, consider the following story: a scientist runs two independent tests to determine the locus of a genetic disease on the human genome. In the first case, the tests respectively point to two fairly narrow regions that just about overlap in a particular region. In the second case, the tests respectively point to fairly broad regions that have minimal overlap in the very same region. Suppose that the tests are halfway reliable and that this region is a somewhat surprising locus for the disease. Then certainly the degree of confidence that the locus of the disease is in this region is greater in the former case, in which the information is more coherent, than in the latter case, in which the information is less coherent.

We define measures for each of these determinants of the degree of confidence in a probabilistic framework. The real challenge lies in developing a measure of coherence (cf. Lewis 1946, Bonjour 1985). This measure defines a partial ordering over information sets. Subsequently, we argue that belief expansion is a function of the reliability of the sources and the coherence of the new information with the information that we already believe. We construct an acceptance measure which determines whether newly acquired information can be added to our beliefs under alternative suppositions about the reliability of the sources. Our calculations rest on some results in the theory of Bayesian Networks. Throughout we have made some strong idealizations. We show how these idealizations can be relaxed by directly invoking Bayesian Networks.

The Model

For each proposition \( R_i \) (in roman script) in the information set, let us define a propositional variable \( R_i \) (in italic script) which can take on two values, viz. \( R_i \) and \( \neg R_i \) (i.e. not-\( R_i \)) for \( i = 1, \ldots, n \). Let \( REPR_i \) be a propositional variable which can take on two values, viz. \( REPR_i \), i.e. there is a report from the proper source to the effect that \( R_i \) is true, and \( \neg REPR_i \), i.e. there is a report to the effect that \( R_i \) is false. We construct a joint probability distribution \( P \) over \( R_1, \ldots, R_n, REPR_1, \ldots, REPR_n \) satisfying the constraint that the sources are independent and less than fully reliable.

We model independence by stipulating that \( P \) respects the following conditional independences:

\[
REPR_i \perp R_j, REPR_j | R_i \text{ for } i \neq j; \quad i, j = 1, 2, \ldots, n
\]

or, in words, \( REPR_i \) is probabilistically independent of \( R_j, REPR_j \), given \( R_i \), for \( i \neq j \) and \( i, j = 1, 2, \ldots, n \).

What this means is that the probability that I will receive a report that \( R_i \) given that \( R_i \) is the case or given that \( R_i \) is not the case, is not affected by any additional information about whether \( R_j \) is the case or whether there is a report to the effect that \( R_j \) is the
case. Each source tunes in on the item of information that it is meant to report on: it may not always provide an accurate report, but its report is not affected by what other sources have to report or by other items of information than the one it reports on (Lewis 1946, Bovens and Olsson 1999).

We define a less-than-fully-reliable source as a source that is better than a randomizer, but short of being a truth-teller and make the simplifying idealization that the information sources are equally reliable. We specify the following two parameters: \( P(\text{REPR}|R_i) = p \) and \( P(\text{REPR}|\bar{R}_i) = q \) for \( i = 1, \ldots, n \). If the information sources are truth-tellers, then \( p = 1 \) and \( q = 0 \), while if they are randomizers, then \( p = q > 0 \). We model less-than-full-reliability by imposing the following constraint on \( P \):

\[
P(\text{REPR}|R_i) = p > q = P(\text{REPR}|\bar{R}_i) > 0 \quad (2)
\]

The degree of confidence in the content of the information set is the posterior joint probability after all the reports have come in:

\[
P^*(R_1, \ldots, R_n) = P(R_1, \ldots, R_n|\text{REPR}_1, \ldots, \text{REPR}_n) \quad (3)
\]

The motivation for the definition of less-than-full reliability is that we are interested in cases in which incoming information raises our confidence in the content of the information set to different levels. When the sources are randomizers, our confidence will be unaffected (Huemer 1997, Bovens and Olsson 1999), i.e. \( P^*(R_1, \ldots, R_n) = P(R_1, \ldots, R_n) \); when they are truth-tellers, our confidence will be raised to certainty, i.e. \( P^*(R_1, \ldots, R_n) = 1 \); and when they are worse than randomizers, our confidence will drop, i.e. \( P^*(R_1, \ldots, R_n) < P(R_1, \ldots, R_n) \).

### Expectation, Reliability and Coherence

It can be shown by the probability calculus, that, given the constraints on \( P \) in \( (1) \) and \( (2) \),

\[
P^*(R_1, \ldots, R_n) = \frac{a_0}{\sum_{i=0}^{n} a_i x^i}, \quad (4)
\]

in which the likelihood ratio \( x = q/p \) (note that \( 0 < x < 1 \) for \( p > q > 0 \)) and \( a_i \) is the sum of the joint probabilities of all combinations of values of the variables \( R_1, \ldots, R_n \) that have \( i \) negative values and \( n - i \) positive values; e.g. for \( n = 3 \),

\[a_2 = P(R_1, \bar{R}_2, \bar{R}_3) + P(\bar{R}_1, R_2, \bar{R}_3) + P(\bar{R}_1, \bar{R}_2, R_3)\]

Note that \( \sum_{i=0}^{n} a_i = 1 \).

We can directly identify the first determinant of the degree of confidence in the information set. Note that \( a_0 = P(R_1, \ldots, R_n) \) is the prior joint probability of the propositions in the information set, i.e. the probability before any information was received. This prior probability is lower for more surprising information and higher for less surprising information. Since more surprising information is tantamount to less expected information, let us call this prior probability the expectation measure. It is easy to see that \( P^*(R_1, \ldots, R_n) \) is a monotonically increasing function of \( a_0 \). We can also directly identify the second determinant, i.e. the reliability of the sources. Note that \( P^*(R_1, \ldots, R_n) \) is a monotonically decreasing function of \( r \) and this measure ranges from 0 for sources that are randomizers to 1 for sources that are truth-tellers.

It is more difficult to construct a coherence measure. Consider the following analogy: to assess the impact of a training program, we consider the rate of the student’s actual performance level over the performance level that he would have reached in an ideal training program, all other things equal. Similarly, to assess the impact of coherence, we consider the rate of the present degree of confidence over the degree of confidence that would have been obtained had the information set been maximally coherent, all other things equal. The information set would have been maximally coherent if and only if \( R_1, \ldots, R_n \) had all been coextensive. Let \( P \) be the actual joint probability distribution. Construct a joint probability distribution \( P_{\text{max}} \) with the same expectation measure and the same reliability measure as \( P \), but \( R_1, \ldots, R_n \) are all coextensive, i.e., on \( P_{\text{max}} \), \( a_0 \) is the same as on \( P \), but \( a_n = 1 - a_0 =: \bar{a}_0 \), so that \( a_i = 0 \), for all \( i \neq 0, n \). It follows from \( (1) \) that,

\[
P_{\text{max}}(R_1, \ldots, R_n) = \frac{a_0}{a_0 + \bar{a}_0 x^n}. \quad (5)
\]

Hence, for \( a_0 \neq 0 \), the ratio

\[
c(x(R_1, \ldots, R_n)) = \frac{P^*(R_1, \ldots, R_n)}{P_{\text{max}}(R_1, \ldots, R_n)} = \frac{a_0 + \bar{a}_0 x^n}{\sum_{i=0}^{n} a_i x^i} \quad (6)
\]

is a measure of the impact of the coherence of the information set on the degree of confidence in the content of the information set. But note that this measure is contingent on the value of the reliability measure: \( P \) only provides us with a reliability-relative coherence measure. This is unwelcome: there is a pretheoretical notion of the coherence of an information set which has nothing to do with the reliability of the sources that provides us with their content. On the other hand, this pretheoretical notion seems to be an ordinal rather than a cardinal notion. And furthermore, it seems to require a partial rather than a complete ordering over information sets: for certain, though not for all pairs of information sets, we are prepared to pass a judgment that one set in the pair is more or less coherent than the other.

It turns out that the reliability-relative coherence measure indeed induces a partial ordering over information sets which is not contingent on the reliability of the sources. Consider two information sets of size \( n \). These sets can be represented by the marginal probability distributions \( P \) and \( P' \) over \( R_1, \ldots, R_n \). It can be shown that for some \( P \) with \( \langle a_0, \ldots, a_n \rangle \) and
with how well the new information coheres with the background information. The more reliable the source is, the less plausible the new information needs to be, given what we already believe, to be justified to add the new information. The more plausible the new information is, given what we already believe, the less reliable the source needs to be, to be justified to add the new information. The challenge is: can a precise account of this relationship be provided?

Our approach is markedly different from AGM belief revision. In the AGM approach, the question is not whether to accept new information or not, but rather, once we have made the decision to accept the new information, how we should revise our beliefs in the face of inconsistency (Makinson 1997, Olsson 1997). Our approach shares a common motivation with the program of non-prioritized belief revision. According to Hansson (1997), we may not be willing to accept the new information because “it may be less reliable (...) than conflicting old information.” Makinson (1997) writes that “we may not want to give top priority to new information (...) we may wish to weigh it against old material, and if it is really just too far-fetched or incredible, we may not wish to accept it.” However, whereas the program of non-prioritized belief revision operates within a logicist framework, we construct a probabilistic model. The cost of this approach is that it is informationally more demanding. The benefit is that it is empirically more adequate, because it is sensitive to degrees of reliability and coherence and to their interplay in belief acceptance. In non-prioritized belief revision, the reliability of the sources does not enter into the model itself and the lack of coherence of an information set is understood in terms of logical inconsistency, which is only a limiting case in our model. To introduce the approach, we address the question of belief expansion. We believe that our model also carries a promise to handle belief revision in general, but this project is beyond the scope of this paper.

We need to make some simplifying assumptions about the origin of the background information and the new information: (a) the propositions in the background information are provided by independent sources, which are (b) less than fully reliable, (c) equally reliable as the new source, and (d) independent of the new source.

Our background information is contained in \(\{R_1, \ldots, R_n\}\). Now suppose that we have a certain threshold level for belief and that the degree of confidence for the background information after having received a report to this effect from independent less than fully reliable sources is right at this level. (This stipulation is not required if we model actual cases by means of Bayesian Networks.) Now we are handed a new item of information \(R_{n+1}\) by an independent less than fully reliable source. Then we will expand our belief set from \(\{R_1, \ldots, R_n\}\) to \(\{R_1, \ldots, R_{n+1}\}\) if and
only if
\[ P(R_1, \ldots, R_{n+1} | \text{REPR}_1, \ldots, \text{REPR}_{n+1}) \geq P(R_1, \ldots, R_n | \text{REPR}_1, \ldots, \text{REPR}_n). \] (7)

Our sources are independent:
\[ \text{REPR}_i \perp R_j, \text{REPR}_j | R_i \quad \text{for } i \neq j; \ i, j = 1, \ldots, n + 1 \]

(8)

We can make the following two observations:

(i) From (10), it is clear that the acceptance measure in terms of the reliability-relative coherence measure \( e \)

\[ e_x(R_1, \ldots, R_m) = P^*(R_1, \ldots, R_m) = \frac{a_0}{\sum_{i=0}^{m} a_i x^i} \] (9)

Considering (6) and (9), we can define this acceptance measure in terms of the reliability-relative coherence measure \( e \), provided that \( a_0 \neq 0 \):

\[ e_x(R_1, \ldots, R_m) = \frac{a_0}{a_0 + \pi_0 x^m} e_x(R_1, \ldots, R_m) \] (10)

From (6) and (10), it follows that we can expand our belief set with a new item of information if and only if

\[ e_x(R_1, \ldots, R_{n+1}) \geq e_x(R_1, \ldots, R_n). \] (11)

We can make the following two observations:

We can formulate a general criterion for belief acceptance: when we have limited knowledge about the reliability of our information sources, we can expand our belief set from \( \{R_1, \ldots, R_n\} \) to \( \{R_1, \ldots, R_{n+1}\} \) if and only if

\[ E(R_1, \ldots, R_{n+1}) \geq E(R_1, \ldots, R_n). \] (13)

### Bayesian Networks

Bayesian Networks represent (conditional) independences between variables and when implemented on a computer they perform complex probabilistic calculations at the touch of a keystroke. We are assuming here that the reader has some familiarity with Bayesian Networks (Cowell et. al. 1999, Jensen 1996, Neapolitan 1990, Pearl 1988).

We construct a Bayesian Network that permits us to read off the reliability-relative coherence measure of an information set \( \{R_1, \ldots, R_n\} \) in Figure 1. First, we construct a Bayesian Network with nodes for the variables \( R_1, \ldots, R_n \) which represents the marginal probability distribution over these variables. Then we add nodes for the variables \( \text{REPR}_1, \ldots, \text{REPR}_n \) and draw an arrow from each node for the variable \( R_i \) to the node for the variable \( \text{REPR}_i \) and specify the conditional probabilities in (2) for each arrow. By the standard criterion of \( d \)-separation, we can now read off the conditional independences in (1) from the network. Subsequently, we construct a node for the variable \( R_1 \& \ldots \& R_n \); we draw in the arrows and specify conditional probabilities such that \( R_1 \& \ldots \& R_n \) holds if and only if \( R_1, \ldots, R_n \) hold. We can now read off \( P^*(R_1, \ldots, R_n) \): it is the probability of \( R_1 \& \ldots \& R_n \) after instantiating \( \text{REPR}_1, \ldots, \text{REPR}_n \). To read off \( P^{\text{max}}(R_1, \ldots, R_n) \), more construction is needed. Notice that \( P^{\text{max}}(R_i) = P^{\text{max}}(R_1 \& \ldots \& R_n) \) for \( i = 1, \ldots, n \) in the counterfactual case of maximal coherence, is equal to \( P(R_i) \& \ldots \& R_n \) in the actual case where the information set may not be maximally coherent. Hence \( P^{\text{max}}(R_1, \ldots, R_n) \) is the posterior joint probability of \( R_1, \ldots, R_n \), had we been informed in the actual case by \( n \) less than fully reliable independent sources that \( R_1 \& \ldots \& R_n \). So we add nodes for the variables \( \text{REPR}, R \) (whose positive values states that
the i-th source informs us that R₁\&\ldots\&Rₙ), draw in the proper arrows and specify the proper conditional probabilities. We can now read off the posterior probabilities. We can now read off the proper arrows and specify the proper conditional probabilities. It is the probability of R₁\&\ldots\&Rₙ being instantiated after instantiating REP₁\&R₂\ldots\&REPₙ\&R. The measure eₓ(R₁\&\ldots\&Rₙ) follows by \([8]\).

We construct a Bayesian Network in Figure 2 to determine whether belief expansion is warranted or not. The construction of the nodes for the variables R₁, \ldots, Rₙ+₁ and REP₁, \ldots, REPₙ\&R should be clear from our construction of the Bayesian Network in Figure 1. This part of the Bayesian Network respects the conditional independences in \([8]\). Now we add a node for the variable R₁\&\ldots\&Rₙ and a node for the variable R₁\&\ldots\&Rₙ\&Rₙ+₁ and specify the conditional probabilities so that R₁\&\ldots\&Rₙ holds if and only if R₁, \ldots, and Rₙ hold and R₁\&\ldots\&Rₙ\&Rₙ+₁ holds if and only if R₁, \ldots, Rₙ, and Rₙ+₁ hold. We instantiate REP₁, \ldots, REPₙ and propagate the evidence throughout the network. We can now read off the acceptance measure eₓ(R₁, \ldots, Rₙ) which is the posterior probability of R₁\&\ldots\&Rₙ. To raise the question of belief expansion, this value should be greater than or equal to our threshold value for belief. Subsequently, we instantiate REPₙ+₁ and propagate the evidence throughout the network. We can now read off the acceptance measure eₓ(R₁, \ldots, Rₙ) which is the posterior probability of R₁\&\ldots\&Rₙ. Depending on our threshold value for belief, we can determine whether we are justified to expand our beliefs with the proposition Rₙ+₁.

It is easy to see how the idealizations can be relaxed in the networks. We can stipulate alternative reliability parameters for the sources. We can add arrows between the REPRᵢ variables or between some REPRᵢ and Rᵢ variables (for i \neq j) to model certain types of dependence between the sources. It suffices that P(R₁, \ldots, Rₙ) is equal to or greater than the threshold value for belief. Furthermore, even if P(R₁, \ldots, Rₙ+₁) is below the threshold value for belief, the model yields a marginal probability distribution over R₁, \ldots, Rₙ. Hence, the general question of belief revision becomes a question of defining a function which maps joint probability distributions over a set of propositional variables into sets of propositions that are values of a subset of these variables and that can reasonably be believed. Defining such a function is beyond the scope of this paper.

**Conclusion**

(i) We have designed a procedure to determine a partial coherence ordering over a set of information sets of size n. If one information set is more coherent than another on this ordering, then our degree of confidence in the content of the former set will be greater than in the content of the latter set, after having been informed by independent and less than fully reliable sources, ceteris paribus. (ii) We have designed a probabilistic criterion for (non-prioritized) belief expansion, which determines whether it is rational to believe new information, considering how reliable the sources are and how well the new information coheres with the old information. (iii) If either the sources are sufficiently reliable or the information set sufficiently large, then the question of belief expansion is largely determined by whether the expanded information set is more coherent than the original information set (provided that there exists an ordering of this pair of information sets), and only marginally by the reliability of the sources. (iv) We have shown how a coherence ordering over information sets can be constructed by means of Bayesian Networks and how belief expansion can be modeled by means of Bayesian Networks in an empirically adequate manner.

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**References.**


